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Minimal sets in monotone and concave skew-product semiflows I: A general theory ☆

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ABSTRACT

The long-term dynamics of a general monotone and concave skew-product semiflow is analyzed, paying special attention to the region delimited from below by the graph of a semicontinuous sub-equilibrium or by a minimal set admitting a flow extension. Different possibilities arise depending on the existence and number or absence of minimal sets strongly above the initially fixed one, as well as on the coexistence or not of bounded and unbounded semiorbits on the region. Previous results are unified and extended, and significative differences with the sublinear case are established. Some scenarios which are impossible in the autonomous or periodic cases may occur in this setting.

1. Introduction

The purpose of this paper is the analysis of the long-term behavior of the semiorbits of a general monotone and concave skew-product semiflow which is continuously differentiable with respect to the state variable. Due to its theoretical and practical interest, this question has been extensively

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studied in the literature: the works of Krasnoselskii [25,26], Selgrade [42], Aronsson and Mellander [5], Johnson [20], Hirsch [13,14], Smith [44], Aiello et al. [1], Freedman and Peng [11], Johnson et al. [21,22], Chueshov [9], Novo et al. [29,31], and Núñez et al. [34], among others, present interesting examples of this type of analysis.

In an abstract and very general framework in which we assume that bounded semiorbits are relatively compact, we unify and extend many of the most significative properties obtained in the papers mentioned above. As it is common in the analysis of nonautonomous dynamical systems, our phase space is a positively invariant closed subset of $\Omega \times X$, where Ω is a compact metric space with a continuous flow and X is a normal strongly ordered Banach space with positive cone X_+. Frequently, in the applications, this setting is associated to differential equations given by a nonautonomous law whose variation with respect to time presents some recurrence properties, such as almost periodicity or an almost automorphic character. The recurrence we assume here is more general: it reduces to the minimality of the base. The set Ω must be considered as the space in which the law moves, and X as the state space. This formulation allows us to combine techniques of topological and differentiable dynamics to study the long-term behavior of the semiorbits.

The analysis initiated here is continued in Núñez et al. [37], where a much more accurate description is obtained in the cases in which the dynamical system is given by a nonautonomous two-dimensional system of differential equations of ordinary, finite-delay or diffusion type.

The case of a monotone and sublinear skew-product semiflow on $\Omega \times X_+$, highly interesting in applied sciences, presents well-known connection points with the concave case. An exhaustive theory for this type of sublinear semiflows is developed by Takáç [45,46], Krause and Ranft [28], Krause and Nussbaum [27], Jiang [16], Arnold and Chueshov [4], Chueshov [9], Zhao [49,50], Jiang and Zhao [17], and Núñez et al. [34–36]. In the present work and in [37], both concerning the concave case, we pay attention to general invariant subsets of $\Omega \times X$: the positivity property usually imposed in the sublinear case is now replaced by the less restrictive assumption of the existence of a semicontinuous subequilibrium or a minimal set, and the region we analyze, at least as starting point, is the region delimited from below (in the fiber-order sense) by it. The scenarios so described are compatible with the presence of almost automorphic extensions of the base not agreeing with copies of Ω, which implies that their dynamics is highly complicated: it presents sensitive dependence with respect to initial conditions and may also admit non-null Lyapunov exponents and several ergodic measures. This is a very significative difference with the sublinear setting: we prove in [36] that such sets cannot exist for different types of nonautonomous differential equations. So that, on the one hand, the analysis of concave semiflows cannot be directly derived from the known properties for the sublinear ones. And, on the other hand, even in the cases in which the restriction of a concave semiflow to an invariant subset of $\Omega \times X$ is also sublinear or can be taken homeomorphically to a sublinear one (i.e., when our starting point is the existence of a continuous equilibrium), we show that the description of a concave semiflow is simpler than for a sublinear one: the number of dynamical possibilities in the concave case is inferior. These facts allow us to assert that the theory developed in these two papers, which respectively follow the storylines of [35] and [36], is at the same time a natural continuation of the one contained there and an independent extension of it.

Let us briefly explain the structure and main results of the paper. Section 2 recalls the basic notions of topological and differentiable dynamics as well as of monotone skew-product semiflows required in the rest of the paper. Different definitions and characterizations of concavity fundamental in our description are also summarized.

In Section 3, under the assumption of the existence of a semicontinuous subequilibrium a and of a minimal set strongly above it, we analyze the long-term dynamics of those semiorbits starting at the graph of a or above it. Note that this region is positively invariant, as a trivial consequence of monotonicity. These semiorbits are all bounded, so that the description of their omega-limit sets and the minimal sets they contain is a fundamental tool to understand their long-term dynamics. Every semiorbit which at a certain time is strongly above the graph of a remains uniformly strongly above it and is uniformly stable, properties which imply that its omega-limit set is a uniformly stable minimal set strongly above a. Any minimal set not contained in one of these omega-limit sets does not contain any point strongly above the subequilibrium. In addition, the omega-limit set of any semiorbit with initial data on the graph of a contains a unique minimal set given by an almost automorphic extension of the base, which does not necessarily agree with a copy of the base and hence, as said before, can present a highly complicated dynamics.

The section is completed with the description of the set $\mathcal{M}_a$ of minimal sets strongly above a. In particular, only two possible scenarios appear for it, instead of the three which may occur in the sublinear case. In the first situation, in which $\mathcal{M}_a$ contains a unique element, this one is a copy of the base which attracts asymptotically all the semiorbits which are eventually strongly above a. In addition, the convergence is exponential, a new difference with the sublinear case, in which this last property does not always hold. We also show that this first scenario holds if the semiflow is eventually strongly concave or if the subequilibrium a is strong. In the opposite situation, the existence of more than one element in $\mathcal{M}_a$ ensures the existence of an infinite quantity of them. More precisely, in this second scenario, fixed any $K^2 \in \mathcal{M}_a$ there is another $K^1 \in \mathcal{M}_a$ strictly below it, and a (not necessarily unique) connected family $(K^\lambda)_{\lambda\in[1,2]}$ of minimal sets "joining" them, with $K^{\lambda_1} < K^{\lambda_2}$ if $\lambda_1 < \lambda_2$. A top minimal set K^+ strongly above a exists if and only if the union of all of them is bounded, in which case it is a new copy of the base and attracts asymptotically the semiorbits starting above it. However, it follows from the previous description that a lowest element of $\mathcal{M}_a$ cannot exist (and this is precisely the third scenario in the sublinear case). Finally, we show that the structure of the set $\mathcal{M}_a$ is much simpler under a more restrictive strong monotonicity property: roughly speaking, all the minimal sets are aligned and constitute a finite or infinite segment in the phase space. This segment may continue "below" a, and it contains all the minimal sets of the semiflow.

The scenario we consider in Section 4 is somehow more general: we assume the existence of a minimal set K admitting a flow extension and study the region above it. As in the sublinear case analyzed in [35], there can occur four very different dynamical situations. In Case A, every semiorbit starting above K is bounded, and every semiorbit starting strongly above K is uniformly stable and remains uniformly strongly above it. In this case, either there is a unique minimal set strongly above K which is an exponentially stable copy of the base (Case A1), or there exist continuous laminations of infinitely many minimal sets in this region (Case A2). In fact Case A agrees with the situation analyzed in Section 3 in the case that K is the graph of a continuous equilibrium, which is the simplest example satisfying the hypotheses of this section. In Case B, every semiorbit starting above K remains bounded but for any pair of points $(\omega, x) \in K$ and (ω, y) with $y \geqslant x$, there is a sequence of times $(t_n) \uparrow \infty$ such that the differences of the corresponding points of the two (ordered) semiorbits belong to $\Omega \times (X_+ - \operatorname{Int} X_+)$. In particular, for any point in another minimal set above K there exists one in K such that the difference among them is not strongly positive. A supplementary condition on strong monotonicity ensures that the omega-limit set of any semiorbit starting above K contains this minimal set and thus it becomes an almost automorphic extension of Ω but not necessarily a copy of it. In addition all the remaining minimal sets are below K. In Case C, there are elements $\omega \in \Omega_b^K \subset \Omega$ with the property that any (ω, x) above K gives rise to a bounded semiorbit, but they do not fill the whole base. The sets Ω_b^K and $\Omega_u^K = \Omega - \Omega_b^K$ are invariant subsets, and Ω_u^K is residual. Similarly to Case B, a condition on strong monotonicity ensures that K is the unique minimal set and is an almost automorphic extension of the base. In addition, under these conditions, there exists an invariant residual subset $\Omega_0^K \subset \Omega$ such that the section of K over $\omega \in \Omega_0^K$ reduces to a unique point (ω, y_ω) and such that any semiorbit (ω, x) with x strongly above y_ω is unbounded but contains sequences of points approaching K. This represents an infinite-dimensional version of the oscillatory behavior obtained in Johnson [18,19] for the solutions of some two-dimensional and scalar almost periodic linear ordinary differential equations. Finally, in Case D, every semiorbit starting strongly above K is unbounded, although it could exhibit again an oscillatory behavior. Well-known examples described in previous works show the completeness of this classification. The section finishes with the analysis of the dynamics below K and its relation with the above one. In particular a complete description of the global dynamics can be obtained in some cases.

2. Basic notions and framework of the problems

A (*real and continuous*) *global flow* on a complete metric space Ω is a continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma(t, \omega)$ satisfying $\sigma_0 = \mathrm{Id}$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t, \omega)$.

By replacing $\mathbb{R}$ by $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$, we obtain the definition of a (*real and continuous*) *global semiflow* on Ω. The flow (resp. semiflow) is *local* if the map σ is defined, continuous, and satisfies the previous properties on an open subset $U \subset \mathbb{R} \times \Omega$ (resp. $U \subset \mathbb{R}_+ \times \Omega$) containing $\{0\} \times \Omega$. We represent by U_ω the set of values of t such that $(t, \omega) \in U$.

The *orbit* of a point ω_0 for a local flow $(\Omega, \sigma, \mathbb{R})$ is $\{\sigma_t(\omega_0) \mid t \in U_{\omega_0}\}$, and it is *globally defined* if $U_{\omega_0} = \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is σ*-invariant* if it is composed by globally defined orbits, and it is *minimal* if in addition it is compact and the orbit of any one of its elements is dense in it. The flow $(\Omega, \sigma, \mathbb{R})$ is *minimal* if Ω is minimal itself.

The (*positive*) *semiorbit* of ω_0 for a local semiflow $(\Omega, \sigma, \mathbb{R}_+)$ is $\{\sigma_t(\omega_0) \mid t \in U_{\omega_0}\}$, and it is *globally defined* if $U_{\omega_0} = \mathbb{R}_+$. A *backward orbit* of ω_0 is a continuous map $\psi : \{-s \mid s \in U_{\omega_0}\} \to \Omega$ such that $\psi(0) = \omega_0$ and for each $s \in U_{\omega_0}$ it is $\sigma(t, \psi(-s)) = \psi(-s + t)$ whenever $0 \leqslant t \leqslant s$. A *flow extension* of $(\Omega, \sigma, \mathbb{R}_+)$ is a flow $(\Omega, \tilde{\sigma}, \mathbb{R})$ such that $\tilde{\sigma}(t, \omega) = \sigma(t, \omega)$ for each $\omega \in \Omega$ and $t \in U_\omega$. A subset Ω_1 of Ω is (*positively*) σ*-invariant* if it is composed by globally defined semiorbits, and it is *minimal* if in addition it is compact and the semiorbit of any one of its elements is dense in it. The *omega-limit set* of a point ω_0 with globally defined and relatively compact semiorbit is given by those points $\omega \in \Omega$ such that $\omega = \lim_{n\to\infty} \sigma(t_n, \omega_0)$ for some sequence $(t_n) \uparrow \infty$. It is a nonempty, compact, connected, positively σ-invariant set, and it contains at least a backward orbit of each one of its points. Note that a minimal set is the omega-limit set of any of its points, and that any omega-limit set contains at least a minimal subset. The semiflow $(\Omega, \sigma, \mathbb{R}_+)$ is *minimal* if Ω is minimal itself. And a compact positively σ-invariant subset Ω_1 *admits a flow extension* if the restricted semiflow does, which as proved by Shen and Yi [43], occurs if every point in Ω_1 admits a unique backward orbit which remains inside the set Ω_1.

The basic properties on topological dynamics here summarized can be found in Ellis [10], Sacker and Sell [40], Shen and Yi [43], and references therein.

As explained in the Introduction, we will be working with skew-product semiflows defined on a bundle whose base is a compact metric space and whose fiber is a strongly ordered Banach space. Some monotonicity and concavity properties will also be imposed. Let us briefly recall these concepts, described e.g. in Amann [2] and Vulikh [47]. From now on $(\Omega, \sigma, \mathbb{R})$ represents a real continuous minimal flow on a compact metric space which we call *the base flow*, we denote $\sigma(t, \omega) = \omega \cdot t$, and X is a Banach space. A real *skew-product semiflow* $(\Omega \times X, \tau, \mathbb{R}_+)$ projecting onto $(\Omega, \sigma, \mathbb{R})$ is a real continuous local semiflow on $\Omega \times X$ of the form

$$\tau : \mathbb{R}_+ \times \Omega \times X \to \Omega \times X, \qquad (t, \omega, x) \mapsto \big(\omega \cdot t, u(t, \omega, x)\big). \tag{2.1}$$

That is, τ preserves the base flow on Ω and is given on the fiber by a map $u : \mathbb{R}_+ \times \Omega \times X \to X$ satisfying the cocycle property

$$u(t + s, \omega, x) = u\big(t, \omega \cdot s, u(s, \omega, x)\big) \quad \text{for } s, t \geqslant 0 \text{ and } (\omega, x) \in \Omega \times X \tag{2.2}$$

whenever all the terms involved are defined. Let K be a minimal subset of $\Omega \times X$. By minimality on the base, for each $\omega \in \Omega$ the section

$$K_\omega = \big\{x \in X \;\big|\; (\omega, x) \in K\big\}$$

contains at least one element. The restricted semiflow $(K, \tau, \mathbb{R}_+)$ is an *almost automorphic extension* of $(\Omega, \sigma, \mathbb{R})$ if there exists $\omega_0 \in \Omega$ such that K_{ω_0} contains a unique element, in which case there is a residual invariant subset of Ω of points with the same property; and K is a *copy of the base* if K_ω reduces to a point for every $\omega \in \Omega$. Clearly, a copy of the base agrees with the graph of a continuous map $c : \Omega \to X$ satisfying $u(t, \omega, c(\omega)) = c(\omega \cdot t)$ for every $t \geqslant 0$ and $\omega \in \Omega$, and the restriction of the semiflow to this set admits a flow extension which reproduces the base flow. We represent this type of minimal set as $K = \{c\}$. A positively invariant compact subset $M \subset \Omega \times X$ is a *pinched set* if the section M_ω reduces to a point for any ω in a residual subset of Ω.

The Banach space X is *strongly ordered* if it contains a closed convex solid cone X_+ with nonempty interior $\operatorname{Int} X_+$. The (partial) *strong order relation* in X is defined by $x \leqslant y \Leftrightarrow y - x \in X_+$; $x < y \Leftrightarrow x \leqslant y$, $x \neq y$; and $x \ll y \Leftrightarrow y - x \in \operatorname{Int} X_+$. The relations $\geqslant$, $>$ and $\gg$ are defined in the obvious way. The positive cone X_+ is *normal* if the norm of the Banach space X is *semimonotone*, i.e., if there is a positive constant $l > 0$ such that

$$\|x\| \leqslant l\|y\| \quad \text{whenever } 0 \leqslant x \leqslant y. \tag{2.3}$$

A norm on X is *monotone* if $l = 1$. The relations $a \leqslant b$, $a < b$, $a \ll b$ for two maps $a, b : \Omega \to X$ are defined in the usual way. Given a map $a : \Omega \to X$ and a set $M \subset \Omega \times X$, we say that M *is above* a and write $M \geqslant a$ if $y \geqslant a(\omega)$ for every $(\omega, y) \in M$; and we say that it *is strongly above* a and write $M \gg a$ if $y \gg a(\omega)$ for every $(\omega, y) \in M$. Similarly, given two minimal sets M and K we say that M *is above* K (resp. *is strongly above* K) and write $M \geqslant K$ (resp. $M \gg K$) if for every $(\omega, x) \in K$ there exists $(\omega, y) \in M$ such that $y \geqslant x$ (resp. $y \gg x$). And we write $M > K$ if $M \geqslant K$ and $M \neq K$. The converse order relations are defined in the symmetric way. As explained at the end of this section, these relations between minimal sets are symmetric and antisymmetric under some additional conditions. Finally, given two (possibly constant) maps $a, v : \Omega \to X$ with $a \leqslant v$, we denote

$$C_a^v = \big\{(\omega, x) \in \Omega \times X \,\big|\, a(\omega) \leqslant x \leqslant v(\omega)\big\}.$$

The skew-product semiflow $(\Omega \times X, \tau, \mathbb{R}_+)$ is *monotone* if

$$u(t, \omega, x) \leqslant u(t, \omega, y) \quad \text{for } \omega \in \Omega \text{ and } x, y \in X \text{ with } x \leqslant y, \tag{2.4}$$

for those values of $t \geqslant 0$ for which both terms are defined; and it is *eventually strongly monotone* if, in addition, there exists $\tilde{t} > 0$ such that, if $x < y$, the previous inequality is strong for $t \geqslant \tilde{t}$.

The skew-product semiflow $(\Omega \times X, \tau, \mathbb{R}_+)$ is *concave* if

$$\begin{aligned} &u\big(t, \omega, \lambda y + (1-\lambda)x\big) \geqslant \lambda u(t, \omega, y) + (1-\lambda)u(t, \omega, x) \\ &\quad \text{for } \omega \in \Omega,\ \lambda \in [0, 1] \text{ and } x, y \in X \text{ with } x \leqslant y \end{aligned} \tag{2.5}$$

for those values of $t \geqslant 0$ for which all the terms are defined; and it is *eventually strongly concave* if, in addition, there exists $\tilde{t} > 0$ such that, if $x < y$, the previous inequality is strong for $t \geqslant \tilde{t}$. It is easy to check that a concave semiflow satisfies

$$\begin{aligned} &u\big(t, \omega, \lambda y + (1-\lambda)x\big) \leqslant \lambda u(t, \omega, y) + (1-\lambda)u(t, \omega, x) \\ &\quad \text{for } \omega \in \Omega,\ \lambda \notin (0, 1) \text{ and } x, y \in X \text{ with } x \leqslant y \end{aligned} \tag{2.6}$$

for those values of $t \geqslant 0$ for which all the terms are defined. And if it is both monotone and concave, it follows from (2.4)–(2.6) that

$$\begin{aligned} &u(t, \omega, z) \geqslant \lambda u(t, \omega, y) + (1-\lambda)u(t, \omega, x) \\ &\quad \text{for } \omega \in \Omega,\ \lambda \in [0, 1] \text{ and } z \geqslant \lambda y + (1-\lambda)x \text{ with } x \leqslant y, \\ &u(t, \omega, z) \leqslant \lambda u(t, \omega, y) + (1-\lambda)u(t, \omega, x) \\ &\quad \text{for } \omega \in \Omega,\ \lambda \notin (0, 1) \text{ and } z \leqslant \lambda y + (1-\lambda)x \text{ with } x \leqslant y \end{aligned} \tag{2.7}$$

for those values of $t \geqslant 0$ for which all the terms are defined.

The skew-product semiflow $(\Omega \times X_+, \tau, \mathbb{R}_+)$ is *sublinear* if

$$u(t, \omega, \lambda x) \geqslant \lambda u(t, \omega, x) \quad \text{for } \omega \in \Omega,\ \lambda \in [0,1] \text{ and } x \in X_+$$

for those values of $t \geqslant 0$ for which all the terms are defined. Note that if τ is concave and $u(t, \omega, 0) \geqslant 0$ for every $\omega \in \Omega$ and $t \geqslant 0$, then its restriction to $\Omega \times X_+$ is well defined and sublinear.

We say that the semiflow τ is C^1 in x if the linear differential operator with respect to x, $u_x : (0, \infty) \times \Omega \times X \to \mathcal{L}(X, X)$, is well defined and continuous whenever $u(t, \omega, x)$ exists, and it satisfies $\lim_{t \to 0^+} u_x(t, \omega, x) y = y$ for every $y \in X$ uniformly for (ω, x) in compact sets. In this case,

$$u_x(t+s, \omega, x) = u_x\big(t, \omega \cdot s, u(s, \omega, x)\big) \circ u_x(s, \omega, x) \tag{2.8}$$

for $t, s \geqslant 0$ whenever the terms involved are defined. In addition, if τ is monotone,

$$u_x(t, \omega, x) e \geqslant 0 \quad \text{for } (\omega, x) \in \Omega \times X \text{ and } e \in X \text{ with } e \geqslant 0 \tag{2.9}$$

for those values of $t > 0$ for which it is defined; and if the semiflow is concave, then

$$\begin{aligned} &u_x(t, \omega, y)(y - x) \leqslant u(t, \omega, y) - u(t, \omega, x) \leqslant u_x(t, \omega, x)(y - x) \\ &\quad \text{for } \omega \in \Omega \text{ and } x, y \in X \text{ with } x \leqslant y \end{aligned} \tag{2.10}$$

for those values of $t > 0$ for which all the terms are defined (see [2]).

We finally recall the concepts of uniform stability and fiber distallity. A positive semiorbit $\{\tau(t, \omega, x) \mid t \geqslant 0\}$ of the skew-product semiflow is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that, if $s \geqslant 0$ and $\|u(s, \omega, x) - y\| \leqslant \delta(\varepsilon)$ for certain $y \in X$, then for $t \geqslant 0$, $\|u(s+t, \omega, x) - u(t, \omega \cdot s, y)\| = \|u(t, \omega \cdot s, u(s, \omega, x)) - u(t, \omega \cdot s, y)\| \leqslant \varepsilon$. A positively invariant compact set $M \subseteq \Omega \times X$ is *uniformly stable* if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $(\omega, y) \in M$ and $(\omega, x) \in \Omega \times X$ satisfy $\|y - x\| \leqslant \delta(\varepsilon)$, then $\|u(t, \omega, y) - u(t, \omega, x)\| \leqslant \varepsilon$ for any $t \geqslant 0$. And a positively τ-invariant set $M \subset \Omega \times X$ is (positively) *fiber-distal* if $\inf_{t>0} \|u(t, \omega, y_1) - u(t, \omega, y_2)\| > 0$ for every $\omega \in \Omega$ and $y_1, y_2 \in M_\omega$ with $y_1 \neq y_2$. In the case that M admits a flow extension, $\inf_{t>0}$ is replaced by $\inf_{t \in \mathbb{R}}$.

Remark 2.1. It is easy to check that if the semiflow is monotone and K and M are minimal sets, then $K \leqslant M$ if and only if $M \geqslant K$. In addition, as proved in Proposition 3.5 in [35], the monotonicity also ensures that a positively fiber-distal minimal set K does not contain any ordered pairs, that is, any two points $(\omega, y_1), (\omega, y_2) \in K$ with $y_1 < y_2$. Consequently, if $K \leqslant M \leqslant K$ for two minimal sets K and M and any of them is fiber-distal, then $K = M$.

3. The dynamics above a semicontinuous subequilibrium

Throughout the rest of the paper, and otherwise indicated, we will work with a minimal real continuous global flow $(\Omega, \sigma, \mathbb{R})$, a strongly ordered Banach space X with a normal cone, and a real continuous skew-product local semiflow $(\Omega \times X, \tau, \mathbb{R}_+)$ satisfying:

(h1) the semiflow is C^1 in x, monotone and concave, and
(h2) any bounded semiorbit is globally defined and relatively compact.

As explained in the Introduction, in this section we describe the dynamics in the region delimited "from below" by the graph of a *semicontinuous subequilibrium*, whose existence constitutes part of the third and fundamental hypothesis we work with. The next definitions and results, based on previous concepts and properties of [3], appear in [29].

Definition 3.1. A map $a : \Omega \to X$ such that $u(t, \omega, a(\omega))$ is defined for any $t \geqslant 0$ and $\omega \in \Omega$ is

(a) an *equilibrium* if $a(\omega \cdot t) = u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geqslant 0$,
(b) a *subequilibrium* if $a(\omega \cdot t) \leqslant u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geqslant 0$, and
(c) a *superequilibrium* if $a(\omega \cdot t) \geqslant u(t, \omega, a(\omega))$ for any $\omega \in \Omega$ and $t \geqslant 0$.

A *semiequilibrium* is either a sub or a superequilibrium. A subequilibrium (resp. superequilibrium) $a : \Omega \to X$ is *semicontinuous* if

(s1) $\Gamma_a = \text{closure}_{\Omega \times X}\{(\omega, a(\omega)) \mid \omega \in \Omega\}$ is a compact subset of $\Omega \times X$, and
(s2) $C_a = \{(\omega, x) \in \Omega \times X \mid x \geqslant a(\omega)\}$ (resp. $C_a = \{(\omega, x) \in \Omega \times X \mid x \leqslant a(\omega)\}$) is a closed subset of $\Omega \times X$.

A subequilibrium (resp. superequilibrium) $a : \Omega \to X$ is *strong* if there exists $s_* > 0$ such that $a(\omega \cdot s_*) \ll u(s_*, \omega, a(\omega))$ (resp. $a(\omega \cdot s_*) \gg u(s_*, \omega, a(\omega))$) for every $\omega \in \Omega$.

Proposition 3.2. *Let $a : \Omega \to X$ be a semicontinuous subequilibrium. Assume that $\lim_{n\to\infty} \omega_n = \omega$ and $a(\omega_n) \leqslant x_n$. Then $a(\omega) \leqslant \lim_{n\to\infty} x_n$ if the last limit exists. In addition, a has a residual set of continuity points, which contains a σ-invariant residual set in the case that a is an equilibrium. Furthermore,*

(i) *for any $t \geqslant 0$, the map $a_t : \Omega \to X$, $\omega \mapsto a_t(\omega) = u(t, \omega \cdot (-t), a(\omega \cdot (-t)))$ is a semicontinuous subequilibrium, and the family (a_t) increases as t does.*
(ii) *If a is strong, then there exist an element $e_* \gg 0$ in X and a time $t_* > 0$ such that $a(\omega) + e_* \leqslant a_{t_*}(\omega)$ for every $\omega \in \Omega$.*
(iii) *If there exist a time $t_0 > 0$ and a point $\omega_0 \in \Omega$ which is simultaneously a continuity point for a and $a \circ \sigma_{t_0}$ such that $a(\omega_0 \cdot t_0) \ll u(t_0, \omega_0, a(\omega_0))$, then a is strong.*

In fact the proof of this result requires the property that strongly ordered initial data give rise to strongly ordered semiorbits, proved below in Proposition 3.4. A symmetric result can be formulated for a semicontinuous superequilibrium. Note that copies of the base and graphs of continuous equilibria are equivalent concepts. We will almost always add to hypotheses (h1) and (h2) this third one:

(h3) there exists a semicontinuous subequilibrium a and a minimal set $K \gg a$.

The operator norm of the linear operator $u_x(t, \omega, x) : X \to X$ is represented by $\|u_x(t, \omega, x)\|$, and the omega-limit set of a point (ω, x) with bounded semiorbit is represented by $\mathcal{O}(\omega, x)$. The first results we prove here, Propositions 3.3, 3.4 and 3.6, contain several basic properties which will often be used throughout the paper. The first one is an adaptation to the monotone setting of Proposition 3.2 in Shen and Yi [43] (stated for strong monotonicity). Part of its (similar) proof is included for the reader's convenience.

Proposition 3.3. *Assume that the semiflow τ is monotone and it satisfies hypothesis* (h2). *Let $(\tilde{\omega}, \tilde{x}) \in \Omega \times X$ have bounded semiorbit and let K be a minimal subset of $M = \mathcal{O}(\tilde{\omega}, \tilde{x})$.*

(i) *If $x \geqslant \tilde{x}$ (resp. $x \leqslant \tilde{x}$) for any $(\tilde{\omega}, x) \in K$, then $K_\omega = \{x_\omega\}$ for any ω in a residual subset Ω_0 of Ω (i.e., K is an almost automorphic extension of the base), and K is the unique minimal subset of M with the initial property.*
(ii) *If $x \geqslant \tilde{x}$ (resp. $x \leqslant \tilde{x}$) for any $(\tilde{\omega}, x) \in M$, then $M_\omega = \{x_\omega\}$ for any ω in a residual subset Ω_0 of Ω (i.e., M is a pinched set), and K is the unique minimal subset of M.*

Proof. (i) Let Ω_0 be the residual set of continuity points of $\Omega \to \mathcal{P}_c(K_X)$, $\omega \mapsto K_\omega$, where the set $\mathcal{P}_c(K_X)$ of closed subsets of the compact set $K_X = \{x \in X \mid \exists \omega \in \Omega \text{ with } (\omega, x) \in K\}$ is endowed with the Hausdorff metric (see Lemma 3.2 in [32]). We fix $\omega \in \Omega_0$ in order to check that K_ω reduces to a point. Let us take $(\omega, y) \in K$ and $(\omega, z) \in \mathcal{O}(\tilde{\omega}, \tilde{x})$ and choose $(t_n) \uparrow \infty$ such that $\lim_{n\to\infty}(\tilde{\omega} \cdot t_n, u(t_n, \tilde{\omega}, \tilde{x})) = (\omega, z)$. Since $y \in K_\omega = \lim_{n\to\infty} K_{\tilde{\omega}\cdot t_n}$, then $(\omega, y) = \lim_{n\to\infty}(\tilde{\omega} \cdot t_n, y_n)$ for a suitable sequence $((\tilde{\omega} \cdot t_n, y_n)) \subset K$. By minimality of K, for any $n \in \mathbb{N}$ there exists $(\tilde{\omega}, x_n) \in K$ with $u(t_n, \tilde{\omega}, x_n) = y_n$. Since, by hypothesis, $x_n \geqslant \tilde{x}$ (resp. $x_n \leqslant \tilde{x}$), we conclude from the monotonicity of τ that $y \geqslant z$ (resp. $y \leqslant z$). This means that y is greater than (resp. less than) any other element in the section of $\mathcal{O}(\tilde{\omega}, \tilde{x})$ over ω. Since y has been arbitrarily chosen in K_ω, we deduce that $K_\omega = \{y\}$, as asserted. This proves the first assertion in (i).

For the second one, let $\widetilde{K} \subseteq \mathcal{O}(\tilde{\omega}, \tilde{x})$ be a minimal set with the initial property assumed on K and take $\omega \in \Omega$ such that $K_\omega = \{y\}$ and $\widetilde{K}_\omega = \{\tilde{y}\}$. The previous proof shows that $y \leqslant \tilde{y} \leqslant y$, so that they agree and hence $(\omega, y) \in K \cap \widetilde{K}$. By minimality, $K = \widetilde{K}$.

(ii) The proof of (ii) follows the same arguments as the previous one. □

Proposition 3.4. *Assume that the semiflow τ is C^1 in x and monotone. Then strongly ordered initial data give rise to strongly ordered semiorbits. If, in addition, τ satisfies hypotheses* (h2)–(h3)*, then there exists a constant vector $e \gg 0$ in X such that $a + e \leqslant K$.*

Proof. Note that, if $\omega \in \Omega$ and $x_0, x_1 \in X$,

$$u(t, \omega, x_1) - u(t, \omega, x_0) = \int_0^1 u_x\big(t, \omega, \lambda x_1 + (1-\lambda)x_0\big)(x_1 - x_0)\, d\lambda \tag{3.1}$$

whenever all the terms involved are defined. We fix $x_0, x_1 \in X$ with $x_1 - x_0 \geqslant e_0 \gg 0$ and $\omega_0 \in \Omega$ and assume that $u(t, \omega_0, x_0)$ and $u(t, \omega_0, x_1)$ are defined for every $t \in [0, T]$. We write $x_\lambda = \lambda x_1 + (1-\lambda)x_0$ for $\lambda \in [0, 1]$. The monotonicity of the semiflow ensures that $u(t, \omega_0, x_0) \leqslant u(t, \omega_0, x_\lambda) \leqslant u(t, \omega_0, x_1)$ for any $\lambda \in [0, 1]$, so that the set $C = \{(\omega_0 \cdot t, u(t, \omega_0, x_\lambda)) \mid t \in [0, T],\ \lambda \in [0, 1]\}$ is well defined. In addition, it is compact: the image by τ of $[0, T] \times \{\omega_0\} \times \{x_\lambda \mid \lambda \in [0, 1]\}$. Since $\lim_{t\to 0^+} u_x(t, \omega, x)e_0 = e_0$ uniformly in $(\omega, x) \in C$, there is $t_0 = t_0(C, e_0) > 0$ such that $u_x(t, \omega, x)e_0 \geqslant (1/2)e_0$ whenever $t \in [0, t_0]$ and $(\omega, x) \in C$. Let us prove that if $n \in \mathbb{N}$, $t \in [0, nt_0] \subset [0, T]$ and $\lambda \in [0, 1]$, then $u_x(t, \omega_0, x_\lambda)e_0 \geqslant (1/2^n)e_0$. It is true for $n = 1$. We assume it for n, take $t \in [0, t_0]$ and apply (2.8), the linearity of $u_x(t, \omega, x)$ and (2.9) to get

$$\begin{aligned} u_x(nt_0 + t, \omega_0, x_\lambda)e_0 &= u_x\big(t, \omega_0 \cdot nt_0, u(nt_0, \omega_0, x_\lambda)\big)u_x(nt_0, \omega_0, x_\lambda)e_0 \\ &\geqslant \big(1/2^n\big)u_x\big(t, \omega_0 \cdot nt_0, u(nt_0, \omega_0, x_\lambda)\big)e_0 \geqslant \big(1/2^{n+1}\big)e_0. \end{aligned}$$

One more iteration ensures that $u_x(T, \omega_0, x_\lambda)e_0 \geqslant (1/2^{n+1})e_0$, with $n = [T/t_0]$. This, (3.1) and again (2.9) guarantee that

$$u(T, \omega_0, x_1) - u(T, \omega_0, x_0) \geqslant k_T e_0 \quad \text{with } k_T > 0, \tag{3.2}$$

proving the first assertion. Note that k_T depends on the choices of ω_0, x_0 and x_1.

We fix a continuity point ω_0 of a, a vector $e_1 \gg 0$ and an open neighborhood U of ω_0 (with closure $\bar{U}$) such that $a(\omega) + e_1 \leqslant x$ whenever $\omega \in \bar{U}$ and $(\omega, x) \in K$. Recall that $u(t, \omega, x)$ is defined for every $t \geqslant 0$ since $(\omega, x) \in K$. We fix $T > 0$. The continuity of τ and property (s1) on a ensure that the set $R = \{(\omega \cdot t, u(t, \omega, \lambda x + (1-\lambda)a(\omega))) \mid t \in [0, T],\ \lambda \in [0, 1],\ \omega \in \bar{U},\ (\omega, x) \in K\}$ is relatively compact in $\Omega \times X$. Repeating for the closure of R the arguments before applied to C, we check the existence of $\kappa_T > 0$ such that

$$u(T, \omega, x) - u\big(T, \omega, a(\omega)\big) \geqslant \kappa_T e_1 \quad \text{for every } (\omega, x) \in K \text{ with } \omega \in \bar{U}. \tag{3.3}$$

By minimality and compactness of Ω, $\Omega = \bigcup_{i=1,\dots,m} U \cdot t_i$ for certain positive times $t_1, \dots, t_m$. We set $\kappa = \min(\kappa_{t_1}, \dots, \kappa_{t_m})$ and define $e = \kappa e_1$. For any $(\omega, x) \in K$ there is $i \in \{1, \dots, m\}$ such that $\omega = \omega_1 \cdot t_i$ for an $\omega_1 \in U$. Since K contains backward orbits of its points, $(\omega, x) = (\omega_1 \cdot t_i, u(t_i, \omega_1, x_1))$ for $(\omega_1, x_1) \in K$. Finally, since a is a subequilibrium, $a(\omega) + e \leqslant u(t_i, \omega_1, a(\omega_1)) + e \leqslant u(t_i, \omega_1, x_1) = x$, which completes the proof. $\square$

Remark 3.5. We fix $e \gg 0$ and assume that $\|u_x(t, \omega, y)e\| \leqslant k$ for (t, ω, y) in a subset U of $\mathbb{R}_+ \times \Omega \times X$. Then, since $\|u_x(t, \omega, y)\| = \sup_{\|w\|=1} \|u_x(t, \omega, y)w\| \leqslant (2l + 1)\lambda_e \|u_x(t, \omega, y)e\|$, where $\lambda_e > 0$ satisfies $-\lambda_e e \leqslant w \leqslant \lambda_e e$ whenever $\|w\| = 1$ and l satisfies (2.3), we conclude that $\|u_x(t, \omega, y)\| \leqslant l_{e,k}$ for any $(t, \omega, y) \in U$, where $l_{e,k} = (2l + 1)\lambda_e k$. Note that $l_{e,k}$ is independent of U.

Proposition 3.6. *Assume that the semiflow satisfies hypotheses* (h1)–(h3)*. Then,*

(i) *any semiorbit starting at C_a is globally defined and bounded. More precisely, given $v \in X$ with $a(\omega) \leqslant v$ for every $\omega \in \Omega$, there exists $k = k(v) > 0$ with*

$$\left\|u(t, \omega, x)\right\| \leqslant k \quad \text{for every } t \geqslant 0 \text{ and } (\omega, x) \in C_a^v. \tag{3.4}$$

(ii) *Given $e \gg 0$ and $v \in X$ with $a(\omega) + e \leqslant v$ for every $\omega \in \Omega$, there exists $k_* = k_*(e, v) > 0$ such that*

$$\left\|u_x(t, \omega, x)\right\| \leqslant k_* \quad \text{for every } t \geqslant 0 \text{ and } (\omega, x) \in C_{a+e}^v. \tag{3.5}$$

(iii) *Given $e \gg 0$ there exists $\lambda \in (0, 1)$ such that $u(t, \omega, x) \geqslant a(\omega \cdot t) + \lambda e$ for any $t \geqslant 0$ whenever $x \geqslant a(\omega) + e$. Consequently, the semiorbit of any (ω, x) with $x \gg a(\omega)$ and its omega-limit set $\mathcal{O}(\omega, x)$ are strongly above a.*

(iv) *The semiorbit of any (ω, x) with $x \gg a(\omega)$ is uniformly stable.*

(v) *If an omega-limit set M is strongly above a, then it is a uniformly stable minimal set admitting a fiber-distal flow extension. In addition, the section map $\Omega \to \mathcal{P}_c(M_X)$, $\omega \mapsto M_\omega = \{x \in X \mid (\omega, x) \in M\}$ is continuous, considering the set $\mathcal{P}_c(M_X)$ of closed subsets of the projection of M over X endowed with the Hausdorff metric.*

(vi) *If $M = \mathcal{O}(\omega, x)$ is strongly above a and such that $x \leqslant z$ for any $(\omega, z) \in M$, or such that $x \geqslant z$ for any $(\omega, z) \in M$, then M is a copy of the base. In particular, if $\mathcal{O}(\omega, a(\omega))$ is strongly above a, then it is a copy of the base.*

Proof. (i) The proof we make is not the simplest one, but it can be adapted to prove (ii). Proposition 3.4 provides $e \gg 0$ with $a(\omega) + e \leqslant y$ for any $(\omega, y) \in K$. Using the monotonicity and concavity properties (2.9) and (2.10) and the fact that a is a subequilibrium, we deduce that, given any $t \geqslant 0$ and $(\omega, y) \in K$,

$$\begin{aligned} 0 \leqslant u_x(t, \omega, y)e &\leqslant u_x(t, \omega, y)\big(y - a(\omega)\big) \leqslant u(t, \omega, y) - u\big(t, \omega, a(\omega)\big) \\ &\leqslant u(t, \omega, y) - a(\omega \cdot t). \end{aligned} \tag{3.6}$$

The boundedness of the invariant set K, condition (s1) on a and the semimonotonicity of the norm hence show the existence of $k_1 > 0$ such that $\|u_x(t, \omega, y)e\| \leqslant k_1$ for every $t \geqslant 0$ and $(\omega, y) \in K$. Remark 3.5 ensures that $\|u_x(t, \omega, y)\| \leqslant l_{e,k_1}$ for $(t, \omega, y) \in \mathbb{R}_+ \times K$. Now take $v \in X$ with $y \leqslant v$ for every $(\omega, y) \in K$. Using again the monotonicity of the semiflow and the concavity property (2.10), $0 \leqslant u(t, \omega, v) - u(t, \omega, y) \leqslant u_x(t, \omega, y)(v - y)$ for every (ω, y) in K whenever $u(t, \omega, v)$ is defined. The semimonotonicity of the norm and the boundedness of K ensure the existence of $k = k(v) > 0$ such that $\|u(t, \omega, v)\| \leqslant k$ whenever it is defined; i.e., for every $t \geqslant 0$ and $\omega \in \Omega$, as (h2) ensures. The bound (3.4) for any $v \geqslant a$ follows from this, the monotonicity of the semiflow and condition (s1) on a.

(ii) Using (i) we can repeat the ideas used in (3.6) in order to show the existence of $k_2 = k_2(e,v) > 0$ such that $\|u_x(t,\omega,x)e\| \leqslant k_2$ for every $t \geqslant 0$ and $(\omega,x) \in C^v_{a+e}$. The conclusion hence follows from Remark 3.5.

(iii) Fix $e \gg 0$ and look for $\lambda_1 \in (0,1)$ such that $a(\omega)+e \geqslant \lambda_1 y + (1-\lambda_1)a(\omega)$ for any $(\omega,y) \in K$. Such a λ_1 exists due to the boundedness of $\{y - a(\omega) \mid (\omega,y) \in K\} \subset X_+$, in turn ensured by condition (s1) on a and the compactness of K. Given (ω,x) with $x \geqslant a(\omega)+e$, we apply (2.7) and have in mind that a is a subequilibrium in order to deduce that $u(t,\omega,x) - a(\omega\cdot t) \geqslant \lambda_1(u(t,\omega,y) - a(\omega\cdot t)) \geqslant \lambda_1\lambda_2 e$ for any $t \geqslant 0$ and $(\omega,y) \in K$, where $\lambda_2 \in (0,1)$ satisfies $a + \lambda_2 e \leqslant K$ (see Proposition 3.4). Hence the semiorbit of (ω,x) lies in a set $C^v_{a+\lambda e}$ for certain $v \in X$ (depending on x) and $\lambda = \lambda_1\lambda_2$, and condition (s2) on a shows that also $\mathcal{O}(\omega,x) \subset C^v_{a+\lambda e}$.

(iv) We fix $\tilde\omega \in \Omega$ and $\tilde x \gg a(\tilde\omega)$. According to (i) and (iii), we can choose $e \gg 0$ and $v \gg 0$ such that $\tau(t,\tilde\omega,\tilde x) \in C^{v/2}_{a+2e}$ for every $t \geqslant 0$. We also choose $\rho > 0$ such that $\|x-y\| < \rho$ ensures that $x - y \leqslant e$ and $y - x \leqslant v/2$: these conditions guarantee that $y - a(\omega) - e \geqslant y - x + e \geqslant 0$ and $v - y \geqslant v/2 + x - y \geqslant 0$ if $(\omega,x) \in C^{v/2}_{a+2e}$ and $\|x-y\| < \rho$. Consequently, if $(\omega,x) \in C^{v/2}_{a+2e}$ and $\|x-y\| < \rho$, then $(\omega,y) \in C^v_{a+e}$. Note that $(\omega, \lambda y + (1-\lambda)x) \in C^v_{a+e}$ for any $(\omega,y),(\omega,x) \in C^v_{a+e}$ and $\lambda \in [0,1]$. The uniform stability hence follows from the representation (3.1) for $\omega = \tilde\omega\cdot s$, $x_1 = y$ and $x_0 = u(s,\tilde\omega,\tilde x)$: given $\varepsilon > 0$, we define $\delta(\varepsilon) = \min(\rho, \varepsilon/k_*)$, with $k_* = k_*(e,v)$ provided by (ii).

(v)–(vi) The proofs of these properties are identical to the ones of Proposition 3.3(iv)–(v) in [35]. □

Note that the minimality of K does not play any role in the proof of Proposition 3.6. By taking it into account, much more can be said about the dynamics on C_a. This is the objective of Theorem 3.8, which provides a description of the set

$$\mathcal{M}_a = \{M \subset \Omega \times X \mid M \text{ is } \tau\text{-minimal with } M \gg a\}.$$

Its proof is based on a new auxiliary result. We say that a continuous map $\gamma : I \to X$ defined on an interval $I \subset \mathbb{R}$ is *strongly increasing* if $\gamma(s_1) \ll \gamma(s_2)$ for $s_1 < s_2$.

Proposition 3.7. *Assume that the semiflow satisfies hypotheses* (h1)–(h3). *Let $\gamma : I \to X$ be a continuous strongly increasing map with $\gamma(\lambda) \gg a(\tilde\omega)$ for every $\lambda \in I$ for a fixed point $\tilde\omega \in \Omega$. Then the map $\Gamma : I \to \mathcal{P}_c(\Omega \times X)$, $\lambda \mapsto K_\lambda = \mathcal{O}(\tilde\omega, \gamma(\lambda))$ is continuous for the Hausdorff topology of the set $\mathcal{P}_c(\Omega \times X)$ of closed parts of $\Omega \times X$. In addition, Γ is monotone, in the sense that $K_{\lambda_1} \leqslant K_{\lambda_2}$ for $\lambda_1 < \lambda_2$; the set $\bigcup_{\lambda\in J} K_\lambda \subset \Omega \times X$ is connected for any subinterval $J \subseteq I$; and Γ is either injective, or constant, or there exists a proper subinterval $J \subset I$ with the same inferior point such that Γ is injective in J and constant in $I - J$.*

Proof. Recall that Proposition 3.6 ensures that $u(t,\tilde\omega,\gamma(\lambda))$ is defined for every $t \geqslant 0$ if $\lambda \in I$, and that the corresponding omega-limit set K_λ is fiber-distal and belongs to $\mathcal{M}_a$. The proof of the properties of monotonicity, continuity and connection is identical to the one of Proposition 3.6(iv) (see also Remark 3.7) in [35]. In order to check the last assertion, we assume that Γ is not injective and take $\lambda_1 < \lambda_2$ in I such that $K_{\lambda_1} = K_{\lambda_2}$ (and hence $K_\lambda = K_{\lambda_1}$ for any $\lambda \in (\lambda_1,\lambda_2)$; see Remark 2.1). We take $\lambda > \lambda_2$ and $(\omega, y_\lambda) \in K_\lambda$. Then $(\omega,y_\lambda) = \lim_{n\to\infty}(\tilde\omega\cdot t_n, u(t_n,\tilde\omega,\gamma(\lambda)))$ for a sequence $(t_n) \uparrow \infty$. By changing to a suitable subsequence if needed, we can assume the existence of $y_{\lambda_i} = \lim_{n\to\infty} u(t_n,\tilde\omega,\gamma(\lambda_i))$ for $i = 1,2$. Since $(\omega, y_{\lambda_i}) \in K_{\lambda_i}$ and, by monotonicity, $y_{\lambda_1} \leqslant y_{\lambda_2}$, it follows from Remark 2.1 that $y_{\lambda_1} = y_{\lambda_2}$. The concavity property (2.10) then shows that $\lim_{n\to\infty} u_x(t_n,\tilde\omega,\gamma(\lambda_2))(\gamma(\lambda_2) - \gamma(\lambda_1)) = 0$, and hence $\lim_{n\to\infty}\|u_x(t_n,\tilde\omega,\gamma(\lambda_2))\| = 0$: see Remark 3.5. Consequently, again by (2.10), $\lim_{n\to\infty}(u(t_n,\tilde\omega,\gamma(\lambda)) - u(t_n,\tilde\omega,\gamma(\lambda_2))) = 0$, and this shows that $(\omega,y_\lambda) \in K_{\lambda_2}$. The minimality of K_{λ_2} shows that $K_{\lambda_2} = K_\lambda$. This means that $L = \{\lambda \in I \mid K_\lambda = K_{\lambda_1}\}$ is a subinterval of I with the same superior. The map Γ is constant if and only if $L = I$; otherwise the last assertion is satisfied by $J = I - L$. □

Theorem 3.8. *Assume that the semiflow satisfies hypotheses* (h1)–(h3). *Then one of the following situations holds*:

Case A1: *K is the unique minimal set strongly above a. In this case, K is a hyperbolic copy of the base, $K = \{c\}$, and it has the following properties*:

(i) *Any semiorbit $u(t,\omega,x)$ with $x \gg a(\omega)$ is globally defined and approaches K asymptotically*: $\lim_{t\to\infty} \|c(\omega\cdot t) - u(t,\omega,x)\| = 0$.
(ii) *The convergence stated in* (i) *is uniform in the sets C^v_{a+e} for $e \gg 0$.*
(iii) *K is the unique positively τ-invariant compact set strongly above a such that any of its points admits a backward orbit inside the set.*
(iv) *For any $v \gg 0$ there exist $\kappa_v > 1$ and $\rho > 0$ such that, if $x \geqslant a(\omega) + v$, then*

$$\bigl\|c(\omega\cdot t) - u(t,\omega,x)\bigr\| \leqslant \kappa_v e^{-\rho t}\bigl\|c(\omega) - x\bigr\| \quad \textit{for } t \geqslant 0.$$

Case A2: *Given any $K^1 \in \mathcal{M}_a$ there exists an $M \in \mathcal{M}_a$ such that $M < K^1$. More precisely, there exists at least a continuous and connected family of minimal sets, $(K^\lambda)_{\lambda\in(0,\infty)}$ (including K^1), such that*

(v) *either $a \ll K^{\lambda_1} < K^{\lambda_2} < K^{\lambda_+} = K^\lambda$ for $0 < \lambda_1 < \lambda_2 < \lambda_+ \leqslant \lambda$ for a $\lambda_+ > 0$, which happens when the union $\mathcal{K}_a \subset \Omega \times X$ of all the minimal sets strongly above a is bounded, in the sense that there exists $e \in X$ such that $x \leqslant e$ for every $(\omega,x) \in \mathcal{K}_a$; or $a \ll K^{\lambda_1} < K^{\lambda_2}$ for $0 < \lambda_1 < \lambda_2$, which happens when $\mathcal{K}_a$ is unbounded. In addition, in the first case, K^{λ_+} is a copy of the base, $K^{\lambda_+} = \{c_{\lambda_+}\}$, and $\lim_{t\to\infty} \|c_{\lambda_+}(\omega\cdot t) - u(t,\omega,x)\| = 0$ whenever $x \geqslant c_{\lambda_+}(\omega)$.*
(vi) *If $(\omega,x) \in C_a$ is such that for any $\lambda \in (0,1]$ there exists $(\omega,y^\lambda) \in K^\lambda$ with $x \leqslant y^\lambda$, then $x \not\gg a(\omega)$.*
(vii) *For any $M \in \mathcal{M}_a$ there is $\lambda \in (0,1)$ such that $K^\lambda < M$.*

Proof. We recall again that Proposition 3.6 ensures that $\mathcal{O}(\omega,x)$ exists and belongs to $\mathcal{M}_a$ whenever $x \gg a(\omega)$.

Case A1. Since $K = \mathcal{O}(\omega,x)$ whenever $x \gg a(\omega)$, it follows from Proposition 3.6(vi) that K is a copy of the base, $\{c\}$: just take (ω,x) with $x \geqslant y$ for any $(\omega,y) \in K$.

(i) Since $\mathcal{O}(\omega,x) = \{c\}$ whenever $x \gg a(\omega)$, the continuity of c proves (i).

(ii) Let us fix $e \gg 0$ and v in X with $a(\omega) + e \leqslant v$ for every $\omega \in \Omega$. In order to prove (ii) there is no restriction in assuming that $a(\omega) + e \leqslant c(\omega) \leqslant v$ for every $\omega \in \Omega$. Then, given any $(\omega,x) \in C^v_{a+e}$, we have

$$u\bigl(t,\omega,a(\omega)+e\bigr) - c(\omega\cdot t) \leqslant u(t,\omega,x) - c(\omega\cdot t) \leqslant u(t,\omega,v) - c(\omega\cdot t),$$

so that the semimonotonicity of the norm ensures that

$$\bigl\|u(t,\omega,x) - c(\omega\cdot t)\bigr\| \leqslant l\bigl\|u(t,\omega,v) - c(\omega\cdot t)\bigr\| + (l+1)\bigl\|c(\omega\cdot t) - u\bigl(t,\omega,a(\omega)+e\bigr)\bigr\|.$$

By (2.9) and (2.10), $0 \leqslant u_x(t,\omega,c(\omega))e/2 \leqslant u(t,\omega,c(\omega)) - u(t,\omega,a(\omega)+e/2)$, and hence we deduce from (i) that $\lim_{t\to\infty} \|u_x(t,\omega,c(\omega))e/2\| = 0$ for every $\omega \in \Omega$. This and Remark 3.5 imply that $\lim_{t\to\infty} \|u_x(t,\omega,c(\omega))\| = 0$ for every $\omega \in \Omega$, which allows us to apply the spectral theory of Chow and Leiva [7,8] in order to conclude the existence of strictly positive constants k_0 and ρ_0 such that

$$\bigl\|u_x\bigl(t,\omega,c(\omega)\bigr)\bigr\| \leqslant k_0 e^{-\rho_0 t} \quad \text{for any } t \geqslant 0 \text{ and } \omega \in \Omega. \tag{3.7}$$

By combining (3.7) with the concavity property (2.10), we see that

$$\lim_{t\to\infty}\bigl\|u(t,\omega,v) - c(\omega\cdot t)\bigr\| = 0 \quad \text{uniformly in } \omega \in \Omega. \tag{3.8}$$

Therefore, the proof of (ii) is complete once checked that

$$\lim_{t\to\infty}\bigl\|c(\omega\cdot t)-u\bigl(t,\omega,a(\omega)+e\bigr)\bigr\|=0 \quad \text{uniformly in } \omega\in\Omega; \tag{3.9}$$

or in other words, that given $\varepsilon>0$ there exists $t(\varepsilon)\geqslant 0$ such that

$$\bigl\|c(\omega\cdot t)-u\bigl(t,\omega,a(\omega)+e\bigr)\bigr\|\leqslant\varepsilon \quad \text{for every } t\geqslant t(\varepsilon) \text{ and } \omega\in\Omega. \tag{3.10}$$

Let us denote $\Gamma_{a+e}=\mathrm{closure}_{\Omega\times X}\{(\omega,a(\omega)+e)\mid\omega\in\Omega\}$ and take $(\omega_0,x_0)\in\Gamma_{a+e}$. Note first that $x_0\leqslant c(\omega_0)$. By the asymptotic behavior before checked, there exists $t_0=t_0(\omega_0,x_0,\varepsilon/k_*)$ such that $\|c(\omega_0\cdot t_0)-u(t_0,\omega_0,x_0)\|<\varepsilon/k_*$, where k_* satisfies the corresponding relation (3.5) for the set $C^{v+e/2}_{a+e/2}$. By the continuity of c and u there exists an open neighborhood of (ω_0,x_0), say $B(\omega_0,x_0)$, such that $\|c(\omega\cdot t_0)-u(t_0,\omega,x)\|<\varepsilon/k_*$ for every $(\omega,x)\in B(\omega_0,x_0)$. By reducing the neighborhood if necessary, we can guarantee that $a(\omega\cdot t_0)+e/2\leqslant u(t_0,\omega,x)\leqslant v+e/2$ for every $(\omega,x)\in B(\omega_0,x_0)$. Hence, for these points, $\|c(\omega\cdot t)-u(t,\omega,x)\|\leqslant\varepsilon$ for every $t\geqslant t_0$, as deduced from the previous properties, the choice of k_*, and representation (3.1). The compactness of Γ_{a+e}, which in turn is deduced from the compactness of Γ_a ensured by condition (s1) on a, allows us to find a finite subcover of this set, say $B(\omega_1,x_1),\dots,B(\omega_n,x_n)$. It is immediate to check that (3.10) holds for $t(\varepsilon)=\max(t_0(\omega_1,x_1,\varepsilon/k_*),\dots,t_0(\omega_n,x_n,\varepsilon/k_*))$.

(iii) Assume the existence of such a set M. We choose $e\gg 0$ and v with $M\subset C^v_{a+e}$, and fix $\varepsilon>0$. Statement (ii) applied to the set C^v_{a+e} provides $t_0>0$ with $\|u(t,\omega,x)-c(\omega\cdot t)\|\leqslant\varepsilon$ for every $t\geqslant t_0$ and every $(\omega,x)\in M$. We now fix $(\omega,x)\in M$ and take $z\in X$ with $(\omega\cdot(-t_0),z)\in M$ and $u(t_0,\omega\cdot(-t_0),z)=x$. Hence $\|x-c(\omega)\|=\|u(t_0,\omega\cdot(-t_0),z)-c((\omega\cdot(-t_0))\cdot t_0)\|\leqslant\varepsilon$, which means that $x=c(\omega)$, and therefore that M and K coincide.

(iv) Obviously it suffices to prove (iv) for a vector $v\gg 0$ with $a(\omega)+2v\leqslant c(\omega)$ for any $\omega\in\Omega$. Let us fix any $k>k_0$ and $0<\rho<\rho_0$, where $k_0\geqslant 1$ and $\rho_0>0$ satisfy (3.7). Let $k_1=l_{v,k}\|v\|$ for the constant $l_{v,k}$ provided by Remark 3.5. We fix a time t_* such that $k_1\leqslant e^{(\rho_0-\rho)t_*}$. The continuity of the map $\mathbb{R}_+\times\Omega\times X\to X$, $(t,\omega,x)\mapsto u_x(t,\omega,x)e$ and relation (3.7) allow us to choose $\delta>0$ such that if $\|c(\omega)-x\|\leqslant\delta$,

$$\bigl\|u_x(s,\omega,x)\bigr\|\leqslant k_1e^{-\rho_0 s} \quad \text{for every } s\in[0,t_*]. \tag{3.11}$$

The uniform stability of K guaranteed by Proposition 3.6(v) provides $\varepsilon\leqslant\delta$ such that if $\|c(\omega)-x\|\leqslant\varepsilon$ then

$$\bigl\|c(\omega\cdot t)-u(t,\omega,x)\bigr\|\leqslant\delta \quad \text{for every } t\geqslant 0, \tag{3.12}$$

so that representation (3.1) for $c(\omega\cdot s)-u(s,\omega,x)$ and relation (3.11) ensure that if $s\in[0,t_*]$,

$$\bigl\|c(\omega\cdot s)-u(s,\omega,x)\bigr\|\leqslant k_1e^{-\rho_0 s}\bigl\|c(\omega)-x\bigr\|. \tag{3.13}$$

Write now any $t\geqslant 0$ as $t=nt_*+s$ with $n\in\mathbb{N}$ and $s\in[0,t_*)$. Relation (3.12) allows us to iterate (3.13) n times for t_* and one more for s in order to obtain

$$\bigl\|c(\omega\cdot t)-u(t,\omega,x)\bigr\|\leqslant k_1^n k_1 e^{-\rho_0 n t_*}e^{-\rho_0 s}\bigl\|c(\omega)-x\bigr\|\leqslant k_1e^{-\rho t}\bigl\|c(\omega)-x\bigr\| \tag{3.14}$$

for any $t\geqslant 0$, as the choice of t_* ensures that $k_1^n\leqslant e^{(\rho_0-\rho)nt_*}$.

Now let us take $(\bar\omega,\bar x)$ with $a(\bar\omega)+v\leqslant\bar x$ and assume first that $\bar x\leqslant c(\bar\omega)$. Let $t_1=t_1(\bar\omega,\bar x)\geqslant 0$ be the minimum time with $\|c(\bar\omega\cdot t_1)-u(t_1,\bar\omega,\bar x)\|\leqslant\varepsilon$. Note that there exists $t(\varepsilon)>0$ independent of $(\bar\omega,\bar x)$ (but depending on v) such that $t_1\leqslant t(\varepsilon)$, as guaranteed by (3.9) and the monotonicity of the semiflow. In the case that $t_1 = 0$, (3.14) holds. Assume now that $t_1 > 0$. Then $\|c(\bar\omega) - \bar x\| > \varepsilon$, and hence $\|c(\bar\omega\cdot t) - u(t,\bar\omega,\bar x)\| \leqslant k_*\|c(\bar\omega) - \bar x\| \leqslant k_* e^{\rho t(\varepsilon)} e^{-\rho t}\|c(\bar\omega) - \bar x\|$, for $0 \leqslant t \leqslant t_1$, where k_* is provided by Proposition 3.6(ii); and for $t \geqslant t_1$, by (3.14),

$$\begin{aligned}\big\|c(\bar\omega\cdot t) - u(t,\bar\omega,\bar x)\big\| &= \big\|u\big(t-t_1,\bar\omega\cdot t_1, c(\bar\omega\cdot t_1)\big) - u\big(t-t_1,\bar\omega\cdot t_1, u(t_1,\bar\omega,\bar x)\big)\big\| \\ &\leqslant k_1 e^{-\rho(t-t_1)}\big\|c(\bar\omega\cdot t_1) - u(t_1,\bar\omega,\bar x)\big\| \leqslant k_1 e^{\rho t_1}\varepsilon e^{-\rho t} \\ &\leqslant k_1 e^{\rho t(\varepsilon)} e^{-\rho t}\big\|c(\bar\omega) - \bar x\big\|.\end{aligned}$$

So that for $\bar x \leqslant c(\bar\omega)$ statement (iv) holds for $\kappa_1 = \max(k_1, k_*,) e^{\rho t(\varepsilon)}$. We point out again that κ_1 depends on v.

The situation is much simpler if $c(\bar\omega) \leqslant \bar x$. In this case, $0 \leqslant u(t,\bar\omega,\bar x) - u(t,\bar\omega,c(\bar\omega)) \leqslant u_x(t,\bar\omega, c(\bar\omega))(\bar x - c(\bar\omega))$, and hence (3.7) ensures that, for $t \geqslant 0$,

$$\big\|c(\bar\omega\cdot t) - u(t,\omega,\bar x)\big\| \leqslant l k_0 e^{-\rho_0 t}\big\|c(\bar\omega) - \bar x\big\| \leqslant l\kappa_1 e^{-\rho t}\big\|c(\bar\omega) - \bar x\big\|.$$

Finally, in the general case, we look for $x_1, x_2 \in X$ with $a(\bar\omega) + v \leqslant x_1 \leqslant \bar x \leqslant x_2$, $x_1 \leqslant c(\bar\omega) \leqslant x_2$ and $\|c(\bar\omega) - x_1\| \leqslant l_1^2\|c(\bar\omega) - \bar x\|$ and $\|c(\bar\omega) - x_2\| \leqslant l_1^2\|c(\bar\omega) - \bar x\|$, where the constant l_1 is independent of the choice of $(\bar\omega, \bar x)$. This can be done by taking

$$\begin{aligned}x_2 &= c(\bar\omega) + \big\|c(\bar\omega) - \bar x\big\|_{c(\bar\omega)-a(\bar\omega)-v}\big(c(\bar\omega) - a(\bar\omega) - v\big), \\ x_1 &= c(\bar\omega) - \min\big(1, \big\|c(\bar\omega) - \bar x\big\|_{c(\bar\omega)-a(\bar\omega)-v}\big)\big(c(\bar\omega) - a(\bar\omega) - v\big),\end{aligned}$$

the constant $l_1 = l_1(v)$ being chosen to satisfy $\|\cdot\| \leqslant l_1\|\cdot\|_{c(\omega)-a(\omega)-v} \leqslant l_1^2\|\cdot\|$ for any $\omega \in \Omega$. (Note that the existence of $w \gg 0$ with $v \leqslant c(\omega) - a(\omega) - v \leqslant w$ for every $\omega \in \Omega$ ensures that $\|\cdot\|_w \leqslant \|\cdot\|_{c(\omega)-a(\omega)-v} \leqslant \|\cdot\|_v$ for every $\omega \in \Omega$.) In this way, the monotonicity of the semiflow ensures that $0 \leqslant u(t,\bar\omega,\bar x) - u(t,\bar\omega,x_1) \leqslant u(t,\bar\omega,x_2) - u(t,\bar\omega,x_1)$. The proof is easily completed by the semimonotonicity of the norm and the previous analysis. This completes the description of Case A1.

Case A2. Assume that Case A1 does not hold. We take a minimal set $K^1 \in \mathcal{M}_a$, fix $(\tilde\omega, \tilde y^1) \in K^1$, consider the continuous strongly increasing map

$$\gamma : (0,\infty) \to X, \qquad \lambda \mapsto \lambda\tilde y^1 + (1-\lambda)a(\tilde\omega), \tag{3.15}$$

and define the map $\Gamma(\lambda) = K^\lambda = \mathcal{O}(\tilde\omega, \gamma(\lambda))$. It is easy to deduce from the monotonicity of the semiflow and Remark 2.1 that Γ is not a constant map: otherwise Case A1 would hold. The family $(K^\lambda)_{\lambda\in(0,\infty)}$, contained in $\mathcal{M}_a$, is monotone, continuous and connected in the sense of Proposition 3.7. Let us check the remaining properties.

(v) It follows from Proposition 3.7 that the non-constant map Γ is not injective if and only if the set K^{λ_+} exists, in which case the injectivity interval of Γ is $(0,\lambda_+]$. Clearly, in this case, the set $\bigcup_{\lambda\in(0,\infty)} K^\lambda$ is bounded, from where it follows easily the boundedness of the set $\mathcal{K}_a$: any $M \in \mathcal{M}_a$ is below a set K^λ for a large enough λ. Proposition 3.6(vi) and the fact that K^{λ_+} agrees with the omega-limit set of $(\tilde\omega, \gamma(\lambda))$ if $\gamma(\lambda) \geqslant y$ for any $(\tilde\omega, y) \in K^{\lambda_+}$ ensure that K^{λ_+} is a copy of the base: $K^{\lambda_+} = \{c_{\lambda_+}\}$. Now we take (ω, x) with $c_{\lambda_+}(\omega) \leqslant x$ and note that monotonicity and the minimality of $\mathcal{O}(\omega, x)$ ensure that $K^{\lambda_+} \leqslant \mathcal{O}(\omega, x) \leqslant K^{\lambda_+}$; hence they agree. In turn this implies the asymptotic behavior stated in (v) in this case.

Finally, assume that $\mathcal{K}_a$ is bounded, with $x \leqslant e$ for every $(\omega, x) \in \mathcal{K}_a$. Take any λ with $\gamma(\lambda) > e$. Proposition 3.6(vi) shows that $\mathcal{O}(\tilde\omega, e)$ is a copy of the base. In addition, on the one hand, by monotonicity, $K^\lambda \geqslant \mathcal{O}(\tilde\omega, e)$; and on the other, since $K^\lambda \subset \mathcal{K}_a$, $K^\lambda \leqslant \mathcal{O}(\tilde\omega, e)$. Hence $\mathcal{O}(\tilde\omega, e) = K^\lambda$. This means that Γ is not injective and hence completes the proof of (v).

(vi) Assume by contradiction the existence of $\omega \in \Omega$ and $x \gg a(\omega)$ such that for any $\lambda \in (0,1]$ there exists $(\omega, y^\lambda) \in K^\lambda$ with $x \leqslant y^\lambda$. Then, by monotonicity, $a \ll \mathcal{O}(\omega, x) \leqslant K^\lambda$ for any $\lambda \in (0,1)$. We take $\lambda > 0$ with $\gamma(\lambda) \leqslant y$ for every $(\tilde{\omega}, y) \in \mathcal{O}(\omega, x)$, and deduce from the monotonicity of the semiflow and the distallity property explained in Remark 2.1 that $K^\lambda = \mathcal{O}(\omega, x)$. According to Proposition 3.7 this implies that Γ is constant, and contradicts Case A2.

(vii) Given $M \in \mathcal{M}_a$, we take $(\tilde{\omega}, x) \in M$ and $\lambda_1, \lambda_2 \in (0, \infty)$ with $\gamma(\lambda_1) \ll \gamma(\lambda_2) \leqslant x$. Then $a \ll K^{\lambda_1} < K^{\lambda_2} \leqslant \mathcal{O}(\tilde{\omega}, x) = M$, as asserted. $\square$

Remarks 3.9. 1. The following property is implicitly contained in the statement of the previous theorem: there exists a *top* minimal set K^+ in $\mathcal{M}_a$, in the sense that $M < K^+$ for any $M \in \mathcal{M}_a$, if and only if the union $\mathcal{K}_a$ of all the minimal sets strongly above a is bounded. The set K^+ agrees with the set K in Case A1 and with K^{λ_+} in Case A2. In addition, K^+ is a copy of the base which attracts asymptotically all the semiorbits starting above it.

In fact, in Case A2, fixed any $K^1 \in \mathcal{M}_a$, the following properties hold for the map γ defined by (3.15) and for $K^\lambda = \mathcal{O}(\tilde{\omega}, \gamma(\lambda))$: a top element of $\mathcal{M}_a$ exists if and only if the injectivity interval J of Proposition 3.7 satisfies $\lambda_+ = \sup J < \infty$, in which case $K^{\lambda_+} = K^+$; if this top minimal set does not exist, for any $M \in \mathcal{M}_a$ there is $\lambda \in (0, \infty)$ with $M < K^\lambda$; and in fact, independently of the existence of K^+, given any minimal set M there exists $\lambda \in J$ such that $M \leqslant K^\lambda$. All these properties are easy consequences of Proposition 3.7, Theorem 3.8 and the monotonicity of the semiflow.

2. There are trivial examples of flows coming from two-dimensional systems of autonomous ordinary differential equations fitting Case A2 and for which there exist infinite different laminations joining two strongly ordered fixed points. Examples 3.13, 3.14 and 3.15 in [37] describe dynamical systems satisfying the conditions of Theorem 3.8 for a non-continuous subequilibrium a, and fitting Cases A2 and A1.

Additional information can be obtained if a point strongly above the graph of a admits a backward orbit which is not strongly above the subequilibrium, as well as under an additional strong-type condition: either on the character of the subequilibrium, or on the concavity, or on the monotonicity. To describe these situations is the objective of the following results.

Theorem 3.10. *Assume that the semiflow satisfies* (h1)–(h3)*. Then,*

(i) *if there exists $(\omega_0, x_0) \in \Omega \times X$ with $x_0 \gg a(\omega_0)$ admitting a local (or global) backward orbit $\{(\omega_0 \cdot s, x_s) \mid -\alpha < s \leqslant 0\}$ for $\alpha > 0$ (or $\alpha = \infty$) such that for every $n \in \mathbb{N}$ there exists s_n with $x_{s_n} \leqslant a(\omega_0 \cdot s_n) + (1/n)\,e$ for an $e \gg 0$ fixed, then the dynamics fits Case* A1 *of Theorem* 3.8*.*

(ii) *If there exist $\omega_1 \in \Omega$ and $t_1 > 0$ such that $a(\omega_1 \cdot t_1) \ll u(t_1, \omega_1, a(\omega_1))$, then the dynamics fits Case* A1 *of Theorem* 3.8*, $\mathcal{O}(\omega_1, a(\omega_1))$ being the unique minimal set strongly above a.*

Proof. (i) Assume by contradiction that the dynamics fits Case A2 of Theorem 3.8, and let $(K^\lambda)_{\lambda \in (0,\infty)}$ be the family appearing in its statement. For each $\lambda > 0$, we take $(\omega_0, y_0^\lambda) \in K^\lambda$ and a backward orbit $\{(\omega_0 \cdot s, y_s^\lambda) \mid s \leqslant 0\} \subset K^\lambda$. Then, for a large enough n, $x_{s_n} \leqslant y_{s_n}^\lambda$. This and the monotonicity mean that $x_0 \leqslant y_0^\lambda$, which according to Theorem 3.8(vi) contradicts $x_0 \gg a(\omega_0)$.

(ii) By hypothesis, $(\omega_0, x_0) = (\omega_1 \cdot t_1, u(t_1, \omega_1, a(\omega_1)))$ satisfies $x_0 \gg a(\omega_0)$. We take its local backward orbit given by $x_s = u(t_1 + s, \omega_1, a(\omega_1))$ for $s \in [-t_1, 0]$, and fix $e \gg 0$. By taking $s_n = -t_1$ for every $n \in \mathbb{N}$, we get $x_{s_n} = a(\omega_1) = a(\omega_0 \cdot s_n) \leqslant a(\omega_0 \cdot s_n) + (1/n)e$. That is, the conditions in (i) are fulfilled. Since $\mathcal{O}(\omega_1, a(\omega_1)) = \mathcal{O}(\omega_1 \cdot t_1, u(t_1, \omega_1, a(\omega_1)))$, Proposition 3.6(iii) and (v) guarantees the last assertion. $\square$

To understand the next statement, recall that Proposition 3.3(ii) ensures the uniqueness and almost automorphic character of a minimal set K contained in the omega-limit set of a point $(\tilde{\omega}, a(\tilde{\omega}))$ in the graph of a semiequilibrium with bounded semiorbit: by monotonicity, $a(\tilde{\omega}) \leqslant x$ for every $(\tilde{\omega}, x) \in \mathcal{O}(\tilde{\omega}, a(\tilde{\omega}))$.

Theorem 3.11. *Assume that the semiflow satisfies hypotheses* (h1) *and* (h2)*. Assume also that it admits a semicontinuous subequilibrium a, and that there exists a point $\tilde{\omega} \in \Omega$ such that the semiorbit of $(\tilde{\omega}, a(\tilde{\omega}))$ is bounded. Let K be the unique minimal subset of $\mathcal{O}(\tilde{\omega}, a(\tilde{\omega}))$. Then*

(1) *either there is a residual subset Ω_0 of Ω such that $K_\omega = \{x_\omega\}$ with $x_\omega - a(\omega) \in X_+ - \operatorname{Int} X_+$ for any $\omega \in \Omega_0$, or*
(2) *a is a strong subequilibrium with $a \ll K$.*

In addition, in case (2)*, the dynamics above a fits Case* A1 *of Theorem* 3.8*, with $K = \mathcal{O}(\omega, a(\omega))$ for any $\omega \in \Omega$. Moreover, the exponentially asymptotic behavior holds in the whole set C_a, that is, if $K = \{c\}$, there exist $\kappa > 1$ and $\rho > 0$ such that, if $(\omega, x) \in C_a$,*

$$\left\| c(\omega \cdot t) - u(t, \omega, x) \right\| \leqslant \kappa e^{-\rho t} \left\| c(\omega) - x \right\| \quad \textit{for } t \geqslant 0.$$

Proof. Note that $a \leqslant K$, since given any $\omega \in \Omega$ there is $(\omega, x) \in K$ and $(t_n) \uparrow \infty$ with $(\omega, x) = \lim_{n\to\infty}(\tilde{\omega} \cdot t_n, u(t_n, \tilde{\omega}, a(\tilde{\omega}))$, so that by semicontinuity and the subequilibrium property, $a(\omega) \leqslant \lim_{n\to\infty} u(t_n, \tilde{\omega}, a(\tilde{\omega})) = x$. Consequently, the semiorbit of any $(\omega, a(\omega))$ is bounded, and its omega-limit set contains a unique minimal set $K_1 \leqslant K$. Interchanging the roles of $\tilde{\omega}$ and ω we conclude that $K \leqslant K_1$, so that they agree, since they are almost automorphic extensions of the base.

Let Ω_0 be the residual subset of Ω given by the intersection of the countable family of residual sets of continuity points of $a \circ \sigma_r$ for $r \in \mathbb{Q}$ (provided by Proposition 3.2 and the fact that σ_r is a homeomorphism) and the residual set for which $K_\omega = \{x_\omega\}$ (provided by the almost automorphic character of K). We take $\omega \in \Omega_0$ and assume that (1) does not hold, i.e., $x_\omega - a(\omega) \gg 0$. Let $(t_n) \uparrow \infty$ satisfy $\lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, a(\omega))) = (\omega, x_\omega)$, with $(t_n) \subset \mathbb{Q}$. Hence $u(t_n, \omega, a(\omega)) \gg a(\omega \cdot t_n)$ for large enough $n \in \mathbb{N}$, since $\lim_{n\to\infty} a(\omega \cdot t_n) = a(\omega)$. Proposition 3.2(iii) ensures that the subequilibrium a is strong.

Let us continue in this situation. According to Proposition 3.2, the semiequilibrium a_t is semicontinuous for any $t > 0$, and there exist a time $t_* > 0$ and an element $v \gg 0$ of X such that $a(\omega) + v \leqslant a_{t_*}(\omega)$ for every $\omega \in \Omega$. We write any $(\omega, y) \in \mathcal{O}(\tilde{\omega}, a(\tilde{\omega}))$ as $(\omega, y) = \lim_{n\to\infty}(\tilde{\omega} \cdot t_n, u(t_n, \tilde{\omega}, a(\tilde{\omega})))$ for some $(t_n) \uparrow \infty$. Since a is a subequilibrium, the monotonicity of τ ensures that

$$a_{t_*}(\tilde{\omega} \cdot t_n) = u\big(t_*, \tilde{\omega} \cdot (t_n - t_*), a\big(\tilde{\omega} \cdot (t_n - t_*)\big)\big) \leqslant u\big(t_n, \tilde{\omega}, a(\tilde{\omega})\big),$$

and hence, using now the semicontinuity of a_{t_*}, we obtain $a(\omega) + v \leqslant a_{t_*}(\omega) \leqslant y$. Therefore, the set $\mathcal{O}(\tilde{\omega}, a(\tilde{\omega}))$ is strongly above a, and so is K: hypotheses (h1)–(h3) are fulfilled.

Since the conditions in Theorem 3.10(ii) are satisfied for any $\omega \in \Omega$, the dynamics above a fits Case A1 of Theorem 3.8 with $K = \mathcal{O}(\omega, a(\omega))$ for any $\omega \in \Omega$.

Finally, to prove that in this case the exponential stability holds in the whole C_a we observe that the uniform convergence to K holds in this case in any set $C_a^w = \{(\omega, x) \in \Omega \times X \mid a(\omega) \leqslant x \leqslant w\}$, since τ takes any point of the graph of a to a point of the graph of a_{t_*} in time t_*, and $a(\omega) + v \leqslant a_{t_*}(\omega)$. This fact allows us to repeat the proof of property (iv) of Theorem 3.8 in Case A1 with $v = 0$. The theorem is proved. □

The previously mentioned Examples 3.13, 3.14 and 3.15 in [37] provide cases of dynamical systems fitting situation (1) in the previous theorem: the subequilibrium a generates an almost automorphic extension of the base which does not agree with a copy of Ω.

The proofs of the next two results are the versions in the concave setting of the ones of Theorems 3.12 and 3.13 in [35], but some significative differences justify their inclusion here.

Theorem 3.12. *Assume that the semiflow satisfies* (h1)–(h3) *and the additional eventually strong concavity condition of existence of $\tilde{t} \geqslant 0$ and $\tilde{\omega} \in \Omega$ such that*

$$u\big(\tilde{t}, \tilde{\omega}, \lambda x + (1-\lambda) a(\tilde{\omega})\big) \gg \lambda u(\tilde{t}, \tilde{\omega}, x) + (1-\lambda) a(\tilde{\omega} \cdot \tilde{t}) \tag{3.16}$$

for $x \gg a(\tilde{\omega})$ and $\lambda \in (0,1)$. Then the dynamics fits Case A1 *of Theorem* 3.8.

Proof. Assume by contradiction the existence of $M \in \mathcal{M}_a$ with $M < K$, where $K \gg a$ is the minimal set provided by (h3). For each $(\tilde\omega, z) \in M$ and each $(\tilde\omega, y) \in K$ with $z < y$ we call $\lambda(z, y) \in \mathbb{R}$ the maximum $\lambda \in \mathbb{R}$ (in fact $\lambda \in (0, 1)$) such that $z \geqslant \lambda y + (1-\lambda)a(\tilde\omega)$. Then we define $\lambda(z) = \sup_{y\in K_{\tilde\omega},\, z<y} \lambda(z, y)$, and $\tilde\lambda = \sup_{z\in M_{\tilde\omega}} \lambda(z)$. The compactness of M and K shows the existence of $(\tilde\omega, \tilde z) \in M$ and $(\tilde\omega, \tilde y) \in K$ with $\tilde z < \tilde y$ such that $\tilde\lambda = \lambda(\tilde z, \tilde y)$. Clearly $\tilde\lambda < 1$, since otherwise $\tilde z = \tilde y$; hence conditions (2.4) and (3.16) ensure that $u(\tilde t, \tilde\omega, \tilde z) \gg \tilde\lambda u(\tilde t, \tilde\omega, \tilde y) + (1-\tilde\lambda)\, a(\tilde\omega \cdot \tilde t)$. Therefore, there exists $\lambda_1 > \tilde\lambda$ such that $u(\tilde t, \tilde\omega, \tilde z) \geqslant \lambda_1 u(\tilde t, \tilde\omega, \tilde y) + (1-\lambda_1)a(\tilde\omega \cdot \tilde t)$. According to (2.7), (2.2) and Definition 3.1(b),

$$u(\tilde t + t, \tilde\omega, \tilde z) \geqslant \lambda_1 u(\tilde t + t, \tilde\omega, \tilde y) + (1-\lambda_1)a\big(\tilde\omega \cdot (\tilde t + t)\big)$$

for every $t > 0$. We choose $(t_n) \uparrow \infty$ such that $\lim_{n\to\infty} \tilde\omega \cdot (\tilde t + t_n) = \tilde\omega$ and such that there exist the limits $\tilde z_1 = \lim_{n\to\infty} u(\tilde t + t_n, \tilde\omega, \tilde z)$ and $\tilde y_1 = \lim_{n\to\infty} u(\tilde t + t_n, \tilde\omega, \tilde y)$. Then, taking limits in the previous inequality for $t = t_n$ and having in mind the semicontinuity of a, we obtain $\tilde z_1 \geqslant \lambda_1 \tilde y_1 + (1-\lambda_1)a(\tilde\omega) \gg \tilde\lambda \tilde y_1 + (1-\tilde\lambda)a(\tilde\omega)$. This contradicts the definition of $\tilde\lambda$. □

Let us finally describe the way in which an additional condition on strong monotonicity along a semiorbit determines the global long-term behavior of the system. This additional condition allows us to describe also the dynamics below the initial subequilibrium.

Theorem 3.13. *Assume that the semiflow satisfies* (h1)–(h3)*, as well as this additional eventually strong monotonicity condition*:

(h4) *There exist $\tilde\omega \in \Omega$ and $\tilde t > 0$, with a continuous at $\tilde\omega \cdot \tilde t$, such that if the points $(\tilde\omega, x)$ and $(\tilde\omega, y)$ respectively belong to minimal sets K_1 and K_2 with $K_1 \geqslant a$, and they have backward orbits $\{(\tilde\omega \cdot s, x_s) \mid s \leqslant 0\}$ and $\{(\tilde\omega \cdot s, y_s) \mid s \leqslant 0\}$ such that for a $\lambda \in [0, 1]$ it is $y_s \geqslant \lambda x_s + (1-\lambda)a(\tilde\omega \cdot s)$ for any $s \leqslant 0$, and $u(t, \tilde\omega, y) > \lambda u(t, \tilde\omega, x) + (1-\lambda)a(\tilde\omega \cdot t)$ for certain $t \geqslant 0$, then $u(t + \tilde t, \tilde\omega, y) \gg \lambda u(t + \tilde t, \tilde\omega, x) + (1-\lambda)a(\tilde\omega \cdot (t + \tilde t))$.*

Assume also that the dynamics fits Case A2 *of Theorem* 3.8*. Then,*

(i) *there exist a continuous equilibrium $\tilde a \geqslant a$ which agrees with a at its continuity points, a continuous equilibrium $c^1 \gg \tilde a$, and a closed (maybe unbounded) interval $J \subset \mathbb{R}$ containing* $[0, 1]$ *such that for any $\lambda \in J$ the graph K^λ of the map $c^\lambda = \lambda c^1 + (1-\lambda)\tilde a$ is minimal. In addition, the sets K^λ for* $\inf J < \lambda \in J$ *are uniformly stable. And any minimal set above a is one of the sets K^λ for $\lambda \in J \cap [0, \infty)$.*
(ii) *If $x \gg a(\omega)$, then there is $\lambda \in J \cap (0, \infty)$ with $\lim_{t\to\infty} \|u(t, \omega, x) - c^\lambda(\omega \cdot t)\| = 0$. More precisely, if $x \geqslant c^{\lambda_0}(\omega)$ for $\lambda_0 > 0$, then $\lambda \geqslant \lambda_0$.*

Assume now that, in addition,

($h4^+$) *hypothesis* (h4) *holds when a is replaced by any continuous equilibrium d and the point $(\tilde\omega, x)$ belongs to a minimal set K_1 with $K_1 \leqslant d$ or $K_1 \geqslant d$.*

Then,

(iii) *any minimal set M agrees with one of the sets K^λ for $\lambda \in J$. In addition, if $c^{\lambda_1} \leqslant M \leqslant c^{\lambda_2}$, then $\lambda \in [\lambda_1, \lambda_2]$.*
(iv) *If $x \geqslant c^{\lambda_0}(\omega)$ for $\lambda_0 >$* $\inf J$*, then $\lim_{t\to\infty} \|u(t, \omega, x) - c^\lambda(\omega \cdot t)\| = 0$ for a $\lambda \geqslant \lambda_0$ in J.*
(v) *If $\lambda_- = \inf J > -\infty$ and $x \ll c^{\lambda_-}(\omega)$, then the semiorbit of (ω, x) is unbounded.*

Finally, assume that (h1)–(h4) *hold and that there are a minimal set $K^1 = \{c^1\} \gg a$, $t_1 > 0$, $\omega_1 \in \Omega$ with a continuous at ω_1 and $\omega_1 \cdot t_1$, and $\lambda_1 \in (0, 1)$ such that*

$$u\big(t_1, \omega_1, \lambda_1 c^1(\omega_1) + (1-\lambda_1)a(\omega_1)\big) > \lambda_1 u\big(t_1, \omega_1, c^1(\omega_1)\big) + (1-\lambda_1)a(\omega_1 \cdot t_1). \tag{3.17}$$

Then the dynamics fits Case A1 *of Theorem* 3.8*.*

Proof. (i) Let us begin by proving, as a preliminary property, that any $M \in \mathcal{M}_a$ is a copy of the base. By the fiber-distal character of M stated in Proposition 3.6(v), it suffices to check that there is a unique element $\tilde{y} \in M_{\tilde{\omega}}$, with $\tilde{\omega}$ given by (h4). Suppose by contradiction the existence of $(\tilde{\omega}, \tilde{y}_1), (\tilde{\omega}, \tilde{y}_2) \in M$ with $\tilde{y}_1 \neq \tilde{y}_2$. For each $t \geqslant 0$, we define $\lambda(t)$ as the maximum real value with

$$u(t, \tilde{\omega}, \tilde{y}_1) \geqslant \lambda(t) u(t, \tilde{\omega}, \tilde{y}_2) + \big(1 - \lambda(t)\big) a(\tilde{\omega} \cdot t).$$

Since, according to Proposition 3.6(v) and Remark 2.1, it cannot be $u(t, \tilde{\omega}, \tilde{y}_1) > u(t, \tilde{\omega}, \tilde{y}_2)$, we have $\lambda(t) \in (0, 1)$; and it follows easily from (2.7), (2.2) and Definition 3.1 that $\lambda(t)$ increases with t. We define $\tilde{\lambda} = \lim_{t\to\infty} \lambda(t) \in (0, 1]$, and take $(t_n) \uparrow \infty$ such that $\lim_{n\to\infty} \tilde{\omega} \cdot t_n = \tilde{\omega}$ and there exist $\lim_{n\to\infty} u(t_n, \tilde{\omega}, \tilde{y}_i) = \tilde{z}_i$ for $i = 1, 2$. Then $(\tilde{\omega}, \tilde{z}_i) \in M$. Let us fix $s \leqslant 0$. According to Proposition 3.6(v), M admits a flow extension, and hence $\lim_{n\to\infty} u(s + t_n, \tilde{\omega}, \tilde{y}_i) = u(s, \tilde{\omega}, \tilde{z}_i)$ for $i = 1, 2$. Taking limits in the previous inequality for $t = s + t_n$ and having in mind the semicontinuity of a, we conclude that, for any $s \leqslant 0$,

$$u(s, \tilde{\omega}, \tilde{z}_1) \geqslant \tilde{\lambda} u(s, \tilde{\omega}, \tilde{z}_2) + (1 - \tilde{\lambda}) a(\tilde{\omega} \cdot s).$$

Now assume by contradiction that the inequality is strict for $s = 0$. Then (h4) ensures that $u(\tilde{t}, \tilde{\omega}, \tilde{z}_1) \gg \tilde{\lambda} u(\tilde{t}, \tilde{\omega}, \tilde{z}_2) + (1 - \tilde{\lambda}) a(\tilde{\omega} \cdot \tilde{t})$; or, in other words, by the cocycle property (2.2) and the continuity of a at $\tilde{\omega} \cdot \tilde{t}$,

$$\lim_{n\to\infty} u(\tilde{t} + t_n, \tilde{\omega}, \tilde{y}_1) \gg \tilde{\lambda} \lim_{n\to\infty} u(\tilde{t} + t_n, \tilde{\omega}, \tilde{y}_2) + (1 - \tilde{\lambda}) \lim_{n\to\infty} a\big(\tilde{\omega} \cdot (\tilde{t} + t_n)\big).$$

This contradicts the fact that $\lambda(\tilde{t} + t_n) \leqslant \tilde{\lambda}$ for n large enough. The conclusion is that $\tilde{z}_1 = \tilde{\lambda} \tilde{z}_2 + (1 - \tilde{\lambda}) a(\tilde{\omega}) \leqslant \tilde{z}_2$, which is impossible, again by Remark 2.1. The proof of our initial assertion is complete.

Let us now fix a minimal set $K^1 = \{c^1\} \in \mathcal{M}_a$ and take $M = \{d\} < K^1$ in $\mathcal{M}_a$. Let λ^* be the maximum real value such that $d(\omega) \geqslant \lambda^* c^1(\omega) + (1 - \lambda^*) a(\omega)$ for all $\omega \in \Omega$. Note that if the inequality is satisfied at one point ω_0, then, by monotonicity, concavity, minimality of Ω, continuity of d and c^1 and semicontinuity of a, it is satisfied at every $\omega \in \Omega$. It is immediate that $\lambda^* \in (0, 1)$. Moreover, we can apply (h4) in order to deduce that $d(\tilde{\omega} \cdot t) = \lambda^* c^1(\tilde{\omega} \cdot t) + (1 - \lambda^*) a(\tilde{\omega} \cdot t)$ for any $t \geqslant 0$: if the strict inequality held at t, we would obtain $d(\tilde{\omega} \cdot (t + \tilde{t})) \gg \lambda^* c^1(\tilde{\omega} \cdot (t + \tilde{t})) + (1 - \lambda^*) a(\tilde{\omega} \cdot (t + \tilde{t}))$, contradicting the definition of λ^*. Consequently, since by (2.7)

$$d(\tilde{\omega} \cdot t) \geqslant \lambda^* c^1(\tilde{\omega} \cdot t) + \big(1 - \lambda^*\big) u\big(t, \tilde{\omega}, a(\tilde{\omega})\big) \geqslant \lambda^* c^1(\tilde{\omega} \cdot t) + \big(1 - \lambda^*\big) a(\tilde{\omega} \cdot t) = d(\tilde{\omega} \cdot t)$$

for any $t \geqslant 0$, we obtain

$$a(\tilde{\omega} \cdot t) = u\big(t, \tilde{\omega}, a(\tilde{\omega})\big) = \frac{d(\tilde{\omega} \cdot t) - \lambda^* c^1(\tilde{\omega} \cdot t)}{1 - \lambda^*}.$$

It is easy to deduce that the set $K^0 = \mathcal{O}(\tilde{\omega}, a(\tilde{\omega}))$ is a copy of the base: it agrees with the graph of the continuous map $\tilde{a} = (d - \lambda^* c^1)/(1 - \lambda^*)$, which consequently is a continuous equilibrium. In addition, it follows from the density of $\{\tilde{\omega} \cdot t \mid t \geqslant 0\}$ that a and $\tilde{a}$ agree at any continuity point of a. By the definition of λ^*, $\tilde{a} \geqslant a$, so that the first assertions in (i) are proved. Note also that $\tilde{a}$ is the unique continuous map agreeing with a at its continuity points, that a minimal set is above a if and only if it is above $\tilde{a}$, and that a and $\tilde{a}$ agree at the positive semiorbit of $\tilde{\omega}$. Therefore, condition (h4) is also valid for $\tilde{a}$ and all the conclusions already obtained hold if we substitute a by $\tilde{a}$ from the beginning, as we do in what follows.

We have in particular shown that $d = \lambda^* c^1 + (1 - \lambda^*) \tilde{a}$. This means that any minimal set strongly above $\tilde{a}$ and below K^1 is given by the graph of one of the continuous maps $c^\lambda = \lambda c^1 + (1 - \lambda) \tilde{a}$ for

$\lambda \in (0,1)$. Let us define $J = \{\lambda \in \mathbb{R} \mid c^\lambda \text{ is a continuous equilibrium}\}$ and denote $K^\lambda = \{c^\lambda\}$ for $\lambda \in J$. It follows from the description of Case A2 made in Theorem 3.8 that $[0,1] \subset J$. Note also that, for $\lambda \notin J$ (in particular $\lambda \notin [0,1]$), c^λ is a continuous superequilibrium if $u(t,\omega,c^\lambda(\omega))$ is globally defined for every $\omega \in \Omega$ (always true for $\lambda > 0$), as easily deduced from (2.7) since $\tilde{a}$ is an equilibrium. Let us now prove that J is an interval. Firstly, assume that $\lambda_1 < 0$ is in J and take $\lambda \in (\lambda_1, 0)$. By monotonicity, the semiorbit starting at $(\omega, c^\lambda(\omega))$ is bounded and hence globally defined. In addition, since $c^\lambda = \mu c^{\lambda_1} + (1-\mu)\tilde{a}$ for $\mu = \lambda/\lambda_1 \in (0,1)$, it follows from (2.7) that c^λ is a subequilibrium; hence it is a continuous equilibrium, and $\lambda \in J$. The same argument shows that if $\lambda_2 > 1$ belongs to J, then $(1,\lambda_2) \subset J$, and hence the assertion is proved.

Clearly J is closed. The uniform stability of K^λ for $\inf J < \lambda \in J$ follows from Proposition 3.6(v) applied to a continuous equilibrium c^{λ_1} for $\inf J < \lambda_1 < \lambda$.

If a minimal set $M \neq \{\tilde{a}\}$ is above a (and hence above $\tilde{a}$), then $M \gg \tilde{a}$: we take $(\tilde{\omega}, y) \in M$ with $y > \tilde{a}(\tilde{\omega})$ and apply (h4) for $x = \tilde{a}(\tilde{\omega})$ and $\lambda = 1$ in order to conclude that $u(\tilde{t},\tilde{\omega},y) \gg \tilde{a}(\tilde{\omega}\cdot\tilde{t})$, so that the assertion follows from Proposition 3.6(iii), since $M = \mathcal{O}(\tilde{\omega}\cdot\tilde{t}, u(\tilde{t},\tilde{\omega},y))$. Consequently, the proof of (i) will be complete once proved that any element M of $\mathcal{M}_a$ agrees with K^λ for a $\lambda \in J \cap [0,\infty)$. We already know it if $M \leqslant K^1$, so that we assume that this is not the case. We write $M = \{d\}$ and define λ_* as the minimum $\lambda \geqslant 0$ with $d(\omega) \leqslant c^\lambda(\omega)$ for every $\omega \in \Omega$. As seen before, the value is common for every $\omega \in \Omega$. Since $M \nleqslant K^1$, then $\lambda_* > 1$. We rewrite the inequality as $c^1 \geqslant (1/\lambda_*)d + (1-1/\lambda_*)\tilde{a}$ and apply (h4) in order to ensure that $d = c^{\lambda_*}$, as asserted.

(ii) Proposition 3.6(iii) and (v) ensures that the set $\mathcal{O}(\omega,x)$ is minimal if $x \gg a(\omega)$, which together with the above property (i) implies the first assertion in (ii). The second one follows immediately from the monotonicity of the semiflow.

(iii) Let M be a minimal set with $M \ll c^{\lambda_0}$ for a $\lambda_0 \in J$. We fix any $\omega \in \Omega$ and take $\lambda < \lambda_0$ such that $x \ll c^\lambda(\omega)$ for a point $(\omega,x) \in M$. By repeating the arguments of Proposition 3.6, we check that $\mathcal{O}(\omega, c^\lambda(\omega))$ is a copy of the base and $\mathcal{O}(\omega, c^\lambda(\omega)) \gg M$. The property that this map c^λ is a superequilibrium is required, since it guarantees that $\mathcal{O}(\omega, c^\lambda(\omega))$ is below the graph of c^λ. Let us represent $\mathcal{O}(\omega, c^\lambda(\omega)) = \{d^\lambda\}$. Hypothesis (h4$^+$) and (i) guarantee that any minimal set between d^λ and c^{λ_0} is the graph of a convex combination of both maps, so that d^λ itself agrees with c^{λ_1} for $\lambda_1 \leqslant \lambda$. In particular, λ_1 belongs to J, so that also λ does and hence $d^\lambda = c^\lambda$. Let us now define $J_M = \{\lambda \leqslant \lambda_0 \mid M \ll c^\lambda\}$, which as just checked is contained in J, and $\lambda_* = \inf J_M$, which hence belongs to J. Then, if $M < K^{\lambda_*}$, we deduce from (h4) for $\lambda = 1$ the existence of a point in the graph of c^{λ_*} strongly above M; consequently, as checked above, $\lambda_* \in J_M$, which is impossible since J_M is obviously left-open. Therefore $M = K^{\lambda_*}$, as asserted.

Assume now that $M < K^{\lambda_0}$ for $\lambda_0 \in J$, which is true for any minimal set excepting the top one; see Remark 3.9.1. We deduce from (h4$^+$) applied to $\lambda = 1$ that $c^{\lambda_0}(\bar{\omega}) \gg \bar{x}$ for a point $(\bar{\omega},\bar{x}) \in M$. As asserted above, this implies that $K^{\lambda_0} = \mathcal{O}(\bar{\omega}, c^{\lambda_0}(\bar{\omega}))$ is strongly above M, and we are again in the situation of the previous paragraph.

(iv) This assertion follows immediately from (i), (ii) and (h4$^+$), with a substituted by c^{λ_0} if $\lambda_0 < 0$, and from the monotonicity of the semiflow.

(v) Assume finally that $x \leqslant c^\lambda(\omega)$ for $\lambda < \lambda_-$ and, by contradiction, that the semiorbit of (ω,x) (and hence that of $(\omega, c^\lambda(\omega))$) is bounded. Since $u(t,\omega,x) \leqslant u(t,\omega,c^\lambda(\omega)) \leqslant c^\lambda(\omega\cdot t)$, $\mathcal{O}(\omega,x)$ remains below the graph of c^λ. This means the existence of a minimal set strongly below K_{λ_-}, impossible by (iii).

The proof of the last assertion of the theorem is immediate. $\square$

4. The dynamics above and below a minimal set

In this last section we analyze the dynamics under the assumption of the existence of a bounded semiorbit, which implies the existence of a minimal set K contained in its omega-limit set. We adapt to the concave case we are considering the results obtained in [35] for the sublinear case, in order to provide a topological description of the different possibilities for the long-term dynamics in the regions above and below K. We begin with the first region by fixing some notation. Given a minimal set $K \subset \Omega \times X$, we represent

$$X_+^K = \{(\omega, x) \in \Omega \times X \mid x = y + v \text{ for some } (\omega, y) \in K \text{ and } v \in X_+\},$$
$$IX_+^K = \{(\omega, x) \in \Omega \times X \mid x = y + e \text{ for some } (\omega, y) \in K \text{ and } e \in \operatorname{Int} X_+\},$$
$$BX_+^K = \{(\omega, x) \in \Omega \times X \mid x = y + w \text{ for some } (\omega, y) \in K \text{ and } w \in X_+ - \operatorname{Int} X_+\}.$$

Note that the monotonicity of τ ensures the positive invariance of X_+^K, one of the regions of the phase space whose dynamics we are interested in describing. In addition, X_+^K and BX_+^K are closed sets, IX_+^K and BX_+^K are not necessarily disjoint, and IX_+^K is not necessarily open. We will work in what follows adding to hypotheses (h1) and (h2) of the previous section the following one:

(h3*) there exists a minimal set $K \subset \Omega \times X$ admitting a flow extension and the semiflow $\tau|_{X_+^K}$ is globally defined.

Recall that the simplest type of minimal set admitting a flow extension is a continuous equilibrium, and that in this case the results of Section 3 apply if there is another minimal set strongly above it. In fact the main idea of the proofs of this section is to define extensions of the semiflow τ to new ones, sublinear and/or concave, to which the previous results of this paper and those of [35] apply. Let us now define the first one of these extensions: an auxiliary skew-product semiflow $\tilde{\tau}^K$ on the bundle $K \times X_+$ preserving the flow τ on the base, namely

$$\begin{aligned}\tilde{\tau}^K(t, \omega, y, v) &= \bigl(\omega \cdot t, u(t, \omega, y), \tilde{u}(t, \omega, y, v)\bigr)\\ &= \bigl(\omega \cdot t, u(t, \omega, y), u(t, \omega, y + v) - u(t, \omega, y)\bigr).\end{aligned} \tag{4.1}$$

It is not hard to check that $\tilde{\tau}^K$ is in fact a semiflow and to derive from the properties of τ that it is well defined, monotone, concave, C^1 in v, and global. Since, in addition, $\tilde{u}(t, \omega, y, 0) = 0$ for $t \geqslant 0$ and $(\omega, y) \in K$, its concavity implies its sublinearity. Note also that the boundedness of $\{\tilde{u}(t, \omega, y, v) \mid t \geqslant 0\}$ is equivalent to that of $\{u(t, \omega, y + v) \mid t \geqslant 0\}$, since K is τ-invariant and bounded, and that bounded $\tilde{\tau}^K$-semiorbits are relatively compact subsets of $K \times X_+$.

Proposition 4.1. *Assume that the semiflow satisfies hypotheses* (h1), (h2) *and* (h3*). *Then $M \subset X_+^K$ is a τ-minimal set if and only if there exists a $\tilde{\tau}^K$-minimal set $\widetilde{M} \subset K \times X_+$ such that*

$$M = \{(\omega, y + v) \mid (\omega, y, v) \in \widetilde{M}\}, \tag{4.2}$$

where $\tilde{\tau}^K$ is given by (4.1)*, and any τ-minimal set $M \geqslant K$ is contained either in BX_+^K or in IX_+^K. In addition,*

(i) *if the semiorbit of a point $(\omega_0, x_1) \in IX_+^K$ is bounded, then so is the semiorbit of any $(\omega_0, x_2) \in X_+^K$.*
(ii) *The sets $\Omega_b^K = \{\omega \in \Omega \mid \sup_{t \geqslant 0} \|u(t, \omega, x)\| < \infty$ whenever $(\omega, x) \in X_+^K\}$ and $\Omega_u^K = \Omega - \Omega_b^K$ are invariant for the base flow.*
(iii) *The set Ω_u^K is either empty or residual in Ω.*

Proof. Assume first that $\widetilde{M}$ is $\tilde{\tau}^K$-minimal and define M by (4.2). We take $(\omega_i, x_i) \in M$ for $i = 1, 2$ and write $x_i = y_i + v_i$ with $(\omega_i, y_i, v_i) \in \widetilde{M}$. By minimality, $(\omega_2, y_2, v_2) = \lim_{n \to \infty}(\omega_1 \cdot t_n, u(t_n, \omega_1, y_1), u(t_n, \omega_1, x_1) - u(t_n, \omega_1, y_1))$ for a suitable $(t_n) \uparrow \infty$, which ensures that $(\omega_2, x_2) = \lim_{n \to \infty}(\omega_1 \cdot t_n, u(t_n, \omega, x_1))$. That is, M agrees with the omega-limit set of any of its points, and hence it is minimal.

Conversely, let M_1 be a τ-minimal set with $M_1 \geqslant K$. If $M_1 = K$, it satisfies the assertion for the continuous $\tilde{\tau}^K$-equilibrium $\widetilde{K} = \{(\omega, y, 0) \mid (\omega, y) \in K\}$. We now assume $M_1 > K$, fix $(\omega, x) \in M_1$, and look for $(\omega, y) \in K$ with $x > y$. Since the $\tilde{\tau}^K$-semiorbit of $(\omega, y, x - y)$ is bounded, its omega-limit set contains a $\tilde{\tau}^K$-minimal set $\widetilde{M}$. We take $(\omega_1, y_1, v_1) \in \widetilde{M}$ and write it as $\lim_{n \to \infty}(\omega \cdot t_n, u(t_n, \omega, y), u(t_n, \omega, x) - u(t_n, \omega, y))$. Then, since $(\omega_1, y_1 + v_1) = \lim_{n \to \infty}(\omega \cdot t_n, u(t_n, \omega, x))$, this point belongs at the same time to M_1 and to the minimal set M defined from $\widetilde{M}$ by (4.2). By minimality of these two sets, M_1 and M agree, as asserted.

The results of [35] ensure that any $\tilde{\tau}^K$-minimal set is contained either in $K \times \operatorname{Int} X_+$ or in $K \times (X_+ - \operatorname{Int} X_+)$, which implies the second assertion of the proposition. Let us now check the remaining properties.

(i) According to Proposition 3.1(ii) in [35], the boundedness of the $\tilde{\tau}^K$-semiorbit of (ω_0, y_0, v_1) with $v_1 \gg 0$ ensures the same property for any (ω_0, y_0, v_2) with $v_2 \geqslant 0$. Recall also that the boundedness of $\{\tilde{u}(t, \omega, y, v) \mid t \geqslant 0\}$ is equivalent to that of $\{u(t, \omega, y + v) \mid t \geqslant 0\}$. Let us write $(\omega_0, x_i) = (\omega_0, y_i + e_i)$, with $(\omega_0, y_i) \in K$ for $i = 1, 2$ and $e_1 \gg 0$, $e_2 \geqslant 0$, and look for $\tilde{e}_1, \tilde{e}_2 \gg 0$ such that $y_1 + \tilde{e}_1 = y_2 + \tilde{e}_2$. From the boundedness of the $\tilde{\tau}^K$-semiorbit of (ω_0, y_1, e_1) and the previous properties we successively obtain the boundedness of those of $(\omega_0, y_1, \tilde{e}_1)$, $(\omega_0, y_2, \tilde{e}_2)$ and (ω_0, y_2, e_2), and deduce the statement.

(ii) Note that (i) ensures that $\omega \in \Omega_b^K$ if and only if there exist $y \in K_\omega$ and $e \gg 0$ such that $\sup_{t \geqslant 0} \|u(t, \omega, y + e)\| < \infty$, in which case the bound holds for any $y \in K_\omega$ and any $e \geqslant 0$. Having this in mind, the proof is similar to the one of Proposition 3.1(iii) in [35]. We take $\omega \in \Omega_b^K$, $y \in K_\omega$, $e \gg 0$ and $s > 0$. By (2.2), $u(s+t, \omega, y+e) = u(t, \omega \cdot s, u(s, \omega, y+e)) = u(t, \omega \cdot s, u(s, \omega, y) + e_1)$, where $e_1 \gg 0$ (see Proposition 3.4), so that $\omega \cdot s \in \Omega_b^K$. Now we take $s < 0$ and $(\omega \cdot s, y_s) \in K$ with $u(-s, \omega \cdot s, y_s) = y$. Then, for $t \geqslant -s$, $u(t, \omega \cdot s, y_s + e) = u(s + t, \omega, u(-s, \omega \cdot s, y_s + e)) = u(s + t, \omega, y + e_2)$ with $e_2 \gg 0$, so that $\omega \cdot s \in \Omega_b^K$. These properties show that Ω_b^K and hence Ω_u^K are invariant for the base flow.

(iii) We assume Ω_u^K is nonempty and fix $e \gg 0$ such that $e \gg y$ for any $(\omega, y) \in K$. It follows from (i) that $\Omega_b^K = \bigcup_{m \in \mathbb{N}} A_m$ with $A_m = \{\omega \in \Omega \mid \|u(t, \omega, e)\| \leqslant m \text{ for every } t \geqslant 0\}$, a closed subset of Ω. Assume that the open set $\operatorname{Int} A_m$ is nonempty for an $m \in \mathbb{N}$. By minimality of Ω, for any $\omega \in \Omega$ there is $t \in \mathbb{R}$ with $\omega \cdot t \in \operatorname{Int} A_m \subset \Omega_b^K$, and, since Ω_b^K is invariant, $\omega \in \Omega_b^K$; i.e., $\Omega = \Omega_b^K$. So that if Ω_u^K is nonempty, it is residual (see [6]). $\square$

Theorem 4.2. *Assume that the semiflow satisfies hypotheses* (h1), (h2) *and* (h3*). *Then one of the four following dynamical possibilities holds for* $\tau|_{X_+^K}$:

Case A: $\Omega = \Omega_b^K$ *and for any* $e \gg 0$ *there exists* $\lambda \in (0, 1)$ *such that* $u(t, \omega, y + e) \geqslant u(t, \omega, y) + \lambda e$ *for any* $t \geqslant 0$ *and any* $(\omega, y) \in K$.
Case B: $\Omega = \Omega_b^K$ *and for any* $e \geqslant 0$ *and any* $(\omega, y) \in K$ *there exists* $(t_n) \uparrow \infty$ *such that there exists* $v = \lim_{n\to\infty}(u(t_n, \omega, y + e) - u(t_n, \omega, y))$ *with* $v \in X_+ - \operatorname{Int} X_+$.
Case C: $\Omega \neq \Omega_b^K$ *and* $\Omega \neq \Omega_u^K$, *in which case* Ω_u^K *is residual.*
Case D: $\Omega = \Omega_u^K$.

In addition,

(i) *in Case* A, *the omega-limit set of any semiorbit starting at* IX_+^K *is a uniformly stable minimal set in* IX_+^K *admitting a fiber-distal flow extension. In addition, either there is a unique minimal set in this region* (*Case* A1), *or there are infinitely many of them* (*Case* A2). *Furthermore, if a top minimal set exists in* X_+^K (*as in Case* A1), *then it is a copy of the base.*
(ii) *In Cases* B, C and D, *any possible minimal set in* X_+^K *is contained in* BX_+^K.
(iii) *In Case* C, *if* $(\omega_1, x_1) \in \mathcal{O}(\omega, x)$ *for an* $(\omega, x) \in X_+^K$ *with* $\omega \in \Omega_b^K$ *and* $\omega_1 \in \Omega_u^K$, *then* $(\omega_1, x_1) \notin IX_+^K$.

Proof. Recall that Proposition 4.1(iii) ensures that a nonempty Ω_u^K is residual. In order to check that Cases A–D exhaust the dynamical possibilities, we only have to check that the situation is described either by A or by B when $\Omega = \Omega_b^K$. Let $\tilde{\tau}^K$ be the semiflow defined by (4.1). According to the results in [35], if every $\tilde{\tau}^K$-semiorbit is bounded (as it happens when $\Omega = \Omega_b^K$), then either there exists a $\tilde{\tau}^K$-minimal set in $K \times \operatorname{Int} X_+$ (Case $\widetilde{\mathrm{A}}$ for $\tilde{\tau}^K$) or any $\tilde{\tau}^K$-minimal set is contained in $K \times (X_+ - \operatorname{Int} X_+)$ (Case $\widetilde{\mathrm{B}}$). And, in Case $\widetilde{\mathrm{A}}$, Proposition 3.6(iii) applied to the semiflow $\tilde{\tau}^K$ and to the zero equilibrium $0: K \to X_+$ shows that given any $e \gg 0$ there exists $\lambda \in (0, 1)$ with $\tilde{u}(t, \omega, y, e) \geqslant \lambda e$ for any $t \geqslant 0$ and $(\omega, y) \in K$. It is clear that these two possibilities respectively correspond to Cases A and B for τ.

For further purposes we point out that Cases C and D respectively correspond to Cases $\widetilde{\mathrm{C}}$ and $\widetilde{\mathrm{D}}$ for $\tilde{\tau}^K$. In the first one, there coexist strongly positive initial states giving rise to bounded and unbounded $\tilde{\tau}^K$-semiorbits, while in the last one the $\tilde{\tau}^K$-semiorbit corresponding to any (ω, y, e) with $(\omega, y) \in K$ and $e \gg 0$ is unbounded.

Let us now check the remaining assertions of the theorem.

(i) Assume that Case $\widetilde{\mathrm{A}}$ holds for $\tilde{\tau}^K$. Proposition 3.6 and Theorem 3.8 applied to this semiflow and its continuous equilibrium $\widetilde{K} = \{(\omega, y, 0) \mid (\omega, y) \in K\}$ prove the following two facts.

First, that a point $(\omega_1, y_1, e_1) \in K \times \operatorname{Int} X_+$ determines a uniformly stable semiorbit strongly above $\widetilde{K}$ and hence that its $\tilde{\tau}^K$-omega-limit set $\widetilde{M}$ is a strongly positive and uniformly stable minimal set admitting a fiber-distal flow extension. It is easy to check that the τ-minimal set M obtained from the above one by (4.2) is the τ-omega-limit set of $(\omega_1, y_1 + e_1)$. Let us check that it is uniformly stable. We fix $e_0 \gg 0$ such that $e \geqslant 2e_0$ for any $(\omega, y, e) \in \widetilde{M}$. We also fix $\varepsilon > 0$ and look for $\delta(\varepsilon) > 0$ such that $(\omega, y, e) \in \widetilde{M}$ and $\|e - v\| < \delta(\varepsilon)$ implies $v \geqslant e_0$ and $\|\tilde{u}(t, \omega, y, e) - \tilde{u}(t, \omega, y, v)\| = \|u(t, \omega, y + e) - u(t, \omega, y + v)\| < \varepsilon$ for any $t \geqslant 0$. Let us take $(\omega, y + e) \in M$ with $(\omega, y, e) \in \widetilde{M}$ and $x \in X$ with $\|y + e - x\| < \delta(\varepsilon)$, which in particular implies that $x - y \geqslant e_0$ and hence that $x = y + v$ with $v \gg 0$. Then $\|u(t, \omega, y + e) - u(t, \omega, x)\| = \|u(t, \omega, y + e) - u(t, \omega, y + v)\| < \varepsilon$ for any $t \geqslant 0$, as asserted. Theorem 3.4 in [32] ensures that M admits a fiber-distal flow extension.

And second, that there are either just one or infinitely many $\tilde{\tau}^K$-minimal sets strongly above $\widetilde{K}$. If there is just one, then it is clear from the first assertion in Proposition 4.1 that there is just one τ-minimal set in IX_+^K, whereas in the second case we know that infinitely many of them are ordered. Then, to see that in the latter case there are also infinitely many τ-minimal sets in IX_+^K it suffices to check that $\widetilde{M}_1 < \widetilde{M}_2$ implies $M_1 < M_2$ for the corresponding τ-minimal sets given by (4.2). So, fix $(\omega, y + v_1) \in M_1$, for $(\omega, y, v_1) \in \widetilde{M}_1$. Then by the definition of order between minimal sets, there exists $(\omega, y, v_2) \in \widetilde{M}_2$ with $v_1 \leqslant v_2$ and actually $v_1 < v_2$, as if not the minimal sets would coincide. Therefore $(\omega, y + v_2) \in M_2$ satisfies $y + v_1 < y + v_2$, so that $M_1 \leqslant M_2$. Besides $M_1 \neq M_2$, as otherwise there would be an ordered pair $(\omega, y + v_1)$, $(\omega, y + v_2)$ in the fiber-distal minimal set M_1, which cannot happen (see Remark 2.1).

Finally, Proposition 3.3(i) ensures that if a top τ-minimal set exists in IX_+^K, then it is an almost automorphic extension of the base Ω, since by monotonicity this minimal set is the omega-limit set of any semiorbit starting above it. Since, as said above, it is uniformly stable, then it is a copy of the base (see e.g. Novo et al. [29, Theorem 5.3]).

(ii) If Case $\widetilde{\mathrm{A}}$ does not hold, any $\tilde{\tau}^K$-minimal set is contained in $K \times (X_+ - \operatorname{Int} X_+)$, which together with Proposition 4.1 proves the assertion.

(iii) According to Proposition 4.1(i), if $(\omega_1, x_1) \in IX_+^K$ with $\omega_1 \in \Omega_u^K$, its semiorbit is unbounded, so that (ω_1, x_1) does not belong to any positively invariant bounded set. $\square$

As in the analysis made in Section 3, an additional condition on strong monotonicity provides a more precise dynamical description. Note that property (h4*) formulated below can be understood as a weak version of our former condition (h4).

Theorem 4.3. *Assume that the semiflow satisfies hypotheses* (h1), (h2) *and* (h3*), *as well as this additional eventually strong monotonicity condition*:

(h4*) *There exists $\tilde{\omega} \in \Omega$ such that, if $x > y$ (or $x < y$) for a point $(\tilde{\omega}, y) \in K$ and a point $(\tilde{\omega}, x)$ belonging to a minimal set M and these two points admit backward orbits $\{(\tilde{\omega} \cdot s, x_s) \mid s \leqslant 0\}$ and $\{(\tilde{\omega} \cdot s, y_s) \mid s \leqslant 0\}$ with $x_s \geqslant y_s$ (or $x_s \leqslant y_s$) for any $s \leqslant 0$, then there exists $\tilde{t} > 0$ such that $u(\tilde{t}, \tilde{\omega}, x) \gg u(\tilde{t}, \tilde{\omega}, y)$ (or $u(\tilde{t}, \tilde{\omega}, x) \ll u(\tilde{t}, \tilde{\omega}, y)$).*

Then,

(i) *if the dynamics on X_+^K doesn't fit Case* A *of Theorem 4.2, then K is the only minimal set in this region. In particular, $K \subseteq \mathcal{O}(\omega, x)$ for every $(\omega, x) \in X_+^K$ with bounded semiorbit.*

(ii) *If the dynamics on X_+^K fits Case* B *or* C, *then K is an almost automorphic extension of the base, and any minimal set M satisfies $M \leqslant K$.*

(iii) *If the dynamics on X_+^K fits Case* C, *then there exists a σ-invariant residual set $\Omega_0^K \subseteq \Omega$ such that K_ω reduces to a point y_ω for any $\omega \in \Omega_0^K$ and such that $\liminf_{t\to\infty} \|u(t,\omega,x) - y_{\omega\cdot t}\| = 0$ and $\limsup_{t\to\infty} \|u(t,\omega,x) - y_{\omega\cdot t}\| = \infty$ whenever $\omega \in \Omega_0^K$ and $x \gg y_\omega$.*
Furthermore, assume that (h4*) *holds for every $\tilde{\omega} \in \Omega$. If $(\omega_0, x_0) \in X_+^K$ and $(\omega_1, x_1) \in X_+^K$ for $\omega_0 \in \Omega_b^K$ and $\omega_1 \in \Omega_u^K$, and $(\omega_1, x_1) \in \mathcal{O}(\omega_0, x_0)$, then $(\omega_1, x_1) \in K$.*

Proof. (i) Let us assume the existence of a minimal set $M \subset X_+^K$ different from K and let $\widetilde{M}$ be a $\tilde{\tau}^K$-minimal set related to M by (4.2). It follows from Proposition 4.1 that given $(\tilde{\omega}, x) \in M$ we can write $x = y + w$ with $(\tilde{\omega}, y, w) \in \widetilde{M}$ and $u(t, \tilde{\omega}, x) - u(t, \tilde{\omega}, y) \in X_+ - \operatorname{Int} X_+$ for any $t \geqslant 0$. In addition, $(\tilde{\omega}, y, w)$ admits a backward orbit in $\widetilde{M}$, which implies the existence of ordered backward orbits for the points $(\tilde{\omega}, x)$ and $(\tilde{\omega}, y)$. This contradicts (h4*) and hence shows the uniqueness of K. The last assertion is trivial.

(ii) Since $K \subseteq \mathcal{O}(\omega, x)$ for (ω, x) with $\omega \in \Omega_b^K$ and $x \geqslant y$ for any $(\omega, y) \in K$, the first property in (ii) follows from Proposition 3.3(i). Let M be a minimal set. We take $\omega \in \Omega_b^K$ and x such that $x \geqslant y$ for any $y \in K_\omega$ and $x \geqslant z$ for any $z \in M_\omega$. Let us choose $(\omega_1, y_1) \in K \subseteq \mathcal{O}(\omega, x)$ and write it as $\lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, x))$ for a suitable $(t_n) \uparrow \infty$. We take $(\omega, z_1) \in M$ and assume without restriction that there exists $(\omega_1, z_2) = \lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, z_1))$. By monotonicity, $z_2 \leqslant y_1$, which means that $M \leqslant K$.

(iii) We fix $e \gg 0$ and define $i_e : K \to [0, \infty)$, $(\omega, x) \mapsto \inf_{t\geqslant 0} \|u(t, \omega, x+e) - u(t, \omega, x)\|$, which is an upper-semicontinuous function in K (see e.g. [6]). It follows from Proposition 3.4 that $i_e(\omega, x) = 0$ if and only if $\liminf_{t\to\infty} \|u(t, \omega, x+e) - u(t, \omega, x)\| = 0$. And in this case, as an easy consequence of the sublinearity of $\tilde{\tau}^K$, $i_v(\omega, x) = 0$ for any $v \gg 0$. We define $i_e^* : \Omega \to [0, \infty)$, $\omega \mapsto \sup_{x\in K_\omega} i_e(\omega, x)$. Let us first check that the compactness of K_ω implies that i_e^* is also upper-semicontinuous. Let $(\omega_n) \subset \Omega$ be a sequence with limit $\omega \in \Omega$ such that there exists $i^* = \lim_{n\to\infty} i_e^*(\omega_n)$. For each $n \in \mathbb{N}$ we take $y_n \in K_{\omega_n}$ with $i_e^*(\omega_n) \leqslant i_e(\omega_n, y_n) + 1/n$, and assume without restriction the existence of $(\omega, y) = \lim_{n\to\infty}(\omega_n, y_n)$. Then $i_e^*(\omega) \geqslant i_e(\omega, y) \geqslant \limsup_{n\to\infty} i_e(\omega_n, y_n) \geqslant \lim_{n\to\infty}(i_e^*(\omega_n) - 1/n) = i^*$, which proves the assertion. Let us now check that the vanishing set Ω_c^K of i_e^* is a σ-invariant residual subset of Ω. As proved by Proposition 3.1(iv) and Theorem 3.14 in [35], the vanishing set of i_e contains any $(\omega, y) \in K$ such that $\{\tilde{u}(t, \omega, y, e) \mid t \geqslant 0\}$ is bounded, and it is τ-invariant. So that $i_e(\omega, y) = 0$ whenever $\omega \in \Omega_b^K$ and $y \in K_\omega$, and hence $i_e^*(\omega) = 0$ for $\omega \in \Omega_b^K$. We now fix $\omega_0 \in \Omega_c^K$ and take $s \in \mathbb{R}$ and $y_s \in K_{\omega_0\cdot s}$. Then $0 = i_e^*(\omega_0) = i_e(\omega_0, u(-s, \omega_0 \cdot s, y_s)) = i_e(\omega_0 \cdot s, y_s)$, so that $i_e^*(\omega_0 \cdot s) = 0$ and Ω_c^K is σ-invariant. To prove that it coincides with the residual set of continuity points of i_e^*, we repeat the argument of Proposition 3.1(iv) in [35].

We now define $\Omega_s^K = \{\omega \in \Omega \mid K_\omega = \{y_\omega\}\} \subseteq \Omega$, a residual subset of Ω which in addition is σ-invariant since K admits a flow extension. The definitions of Ω_c^K and Ω_u^K and Proposition 4.1 guarantee the stated properties for any element ω of the σ-invariant residual set $\Omega_0^K = \Omega_u^K \cap \Omega_c^K \cap \Omega_s^K$.

The last assertion is proved as the analogous one in Theorem 3.15 in [35]. $\square$

There are trivial examples of autonomous scalar equations given by concave functions (and hence giving rise to a concave semiflow) and fitting Cases A1, A2, B and D. The reader is referred to [35] for examples of Case C (which cannot be autonomous). There, also nonautonomous examples of Cases A, B and D with complicated characteristics not possible in the autonomous case are described. They are based on previous results of Furstenberg [12], Sacker and Sell [38,39], Selgrade [41], Johnson [18,20], Novo and Obaya [30], Keller [24], Yi [48], Jäger [15], Jorba et al. [23], and Núñez and Obaya [33].

We complete the paper with a short analysis of the dynamics below the minimal set K, i.e., on the set $X_-^K = \{(\omega, x) \in \Omega \times X \mid x = y - v$ for some $(\omega, y) \in K$ and $v \in X_+\}$, which is again invariant due to monotonicity, and with its relation with the dynamics in the above region.

Proposition 4.4. *Assume that the semiflow satisfies hypotheses* (h1), (h2) *and* (h3*) *and that the restriction of the semiflow to X_-^K is global.*

(i) *The sets* $\widetilde{\Omega}_b^K=\{\omega\in\Omega\mid \exists y\in K_\omega,\ e\gg 0 \text{ with } \sup_{t\geqslant 0}\|u(t,\omega,y-e)\|<\infty\}$ *and* $\widetilde{\Omega}_u^K=\Omega-\widetilde{\Omega}_b^K$ *are invariant for the base flow.*
(ii) $\widetilde{\Omega}_b^K\subseteq\Omega_b^K$ *and* $\Omega_u^K\subseteq\widetilde{\Omega}_u^K$.
(iii) *The set* $\widetilde{\Omega}_u^K$ *is either empty or residual in* Ω.

Proof. (i) We fix $\omega\in\widetilde{\Omega}_b^K$, $y\in K_\omega$ and $e\gg 0$ with $\sup_{t\geqslant 0}\|u(t,\omega,y-e)\|<\infty$, and $s>0$. Then $u(s+t,\omega,y-e)=u(t,\omega\cdot s,u(s,\omega,y-e))=u(t,\omega\cdot s,u(s,\omega,y)-e_1)$, where $e_1\gg 0$ (see Proposition 3.4), so that $\omega\cdot s\in\widetilde{\Omega}_b^K$. Now we take $s<0$, $(\omega\cdot s,y_s)\in K$ with $u(-s,\omega\cdot s,y_s)=y$ and $e_1\gg 0$ such that $0\ll u(-s,\omega\cdot s,y_s)-u(-s,\omega\cdot s,y_s-e_1)\leqslant e$. Then, for $t\geqslant -s$, $u(t,\omega\cdot s,y_s)\geqslant u(t,\omega\cdot s,y_s-e_1)=u(s+t,\omega,u(-s,\omega\cdot s,y_s-e_1))\geqslant u(s+t,\omega,y-e)$, so that $\omega\cdot s\in\Omega_b^K$. Hence, $\widetilde{\Omega}_b^K$ and $\widetilde{\Omega}_u^K$ are invariant.

(ii) Assume that the semiorbit of $(\omega,y-e)$ is bounded for a point $(\omega,y)\in K$ and $e\gg 0$. The concavity of the semiflow ensures that $2u(t,\omega,y)\geqslant u(t,\omega,y-e)+u(t,\omega,y+e)$, which together with the monotonicity implies that $u(t,\omega,y)\leqslant u(t,\omega,y+e)\leqslant 2u(t,\omega,y)-u(t,\omega,y-e)$. The semimonotonicity of the norm ensures the boundedness of the semiorbit of $(\omega,y+e)$, which by Proposition 4.1(i) implies that $\omega\in\Omega_b^K$. The second property in (ii) is a trivial consequence.

(iii) Let us fix $e\gg 0$. It is easy to check that $\widetilde{\Omega}_b^K=\bigcup_{n,m\in\mathbb{N}}A_{n,m}$ with $A_{n,m}=\{\omega\in\Omega\mid\exists y\in K_\omega \text{ with } \|u(t,\omega,y-(1/n)e)\|\leqslant m \text{ for every } t\geqslant 0\}$, closed. The rest of the proof is identical to the one of Proposition 4.1(iii). □

It is possible to find examples showing that the case $\Omega=\widetilde{\Omega}_u^K$ is compatible with Case A, B, C or D above K. On the contrary, as our last results show, a nonempty $\widetilde{\Omega}_b^K$ provides some restrictions on X_+^K.

Theorem 4.5. *Assume that the semiflow satisfies hypotheses* (h1), (h2) *and* (h3*) *and that the restriction of the semiflow to* X_-^K *is global. Assume also that* $\widetilde{\Omega}_b^K$ *is nonempty.*

(i) *Assume that there exist* $(\omega_1,y_1)\in K$, $e\gg 0$ *and* $\delta>0$ *such that the semiorbit of* (ω_1,y_1-e) *is bounded and* $u(t,\omega_1,y_1)-u(t,\omega_1,y_1-e)\geqslant\delta e$ *for any* $t\geqslant 0$. *Then* $\widetilde{\Omega}_b^K=\Omega_b^K=\Omega$.
(ii) *Assume that the condition in* (i) *does not hold and that* (h4*) *holds. Then* K *is the unique minimal set and the dynamics in* X_+^K *fits Case* B *or* C.

Proof. (i) We fix a minimal set $K_1\subseteq\mathcal{O}(\omega_1,y_1-e)$ and consider the skew-product semiflow $\tau^{K_1}:\mathbb{R}_+\times K_1\times X\to K_1\times X$, $(t,\omega,z,x)\mapsto(\omega\cdot t,u(t,\omega,z),u(t,\omega,x))$, which is well defined no matter the fact that the base (K_1,τ) is not a flow but a semiflow. It is easy to check that it satisfies hypotheses (h1) and (h2). Let us check that it also satisfies (h3). First, $k_1:K_1\to X$, $(\omega,z)\mapsto z$ is a continuous equilibrium with graph $\{(\omega,z,z)\mid(\omega,z)\in K_1\}$. Now take $(\tilde\omega,\tilde z)\in K_1$ and look for a suitable sequence $(t_n)\uparrow\infty$ with $\tilde\omega=\lim_{n\to\infty}\omega_1\cdot t_n$, $\tilde z=\lim_{n\to\infty}u(t_n,\omega_1,y_1-e)$, and for which there exists $\tilde y=\lim_{n\to\infty}u(t_n,\omega_1,y_1)$. It follows from the condition in (i) that $u(s,\tilde\omega,\tilde y)-u(s,\tilde\omega,\tilde z)\geqslant\delta e$ for any $s>0$, which implies that the τ^{K_1}-omega-limit set of $(\tilde\omega,\tilde z,\tilde y)$ is strongly above k_1.

Under these conditions, Proposition 3.6(i) (whose arguments do not require a flow on the base) ensures that every τ^{K_1}-semiorbit starting above k_1 is bounded. In other words, the τ-semiorbit of any point (ω,x) with $x\geqslant z$ for some $(\omega,z)\in K_1$ is bounded. This proves (i) since, by construction, $K\gg K_1$.

(ii) We now consider the monotone and concave skew-product semiflow $\tau^K:\mathbb{R}_+\times K\times X\to K\times X$, $(t,\omega,y,x)\mapsto(\omega\cdot t,u(t,\omega,y),u(t,\omega,x))$, for which the set $K_*=\{(\omega,y,y)\mid(\omega,y)\in K\}$ is a copy of the base. As a first step, we will show the absence of a τ^K-minimal set $M_*<K_*$, by assuming by contradiction the existence of such a set. Assume also the existence of $(\omega,y,z)\in M_*$ with $z\ll y$. Due to the condition in (ii), we find $(t_n)\uparrow\infty$ such that $\lim_{n\to\infty}\tau^K(t_n,\omega,y,z)=(\omega_0,y_0,z_0)\in M_*$ with $y_0-z_0\in X_+-\operatorname{Int}X_+$. The first property in Proposition 3.4 ensures that any backward τ^K-orbit $\{(\omega_0\cdot s,y_s,z_s)\mid s\leqslant 0\}\subset M_*$ of (ω_0,y_0,z_0) satisfies $y_s-z_s\in X_+-\operatorname{Int}X_+$. Since this set is closed, the same happens with the set of points obtained as $\lim_{n\to\infty}(\omega_0\cdot s_n,y_{s_n},z_{s_n})$ for $(s_n)\downarrow-\infty$, which, by minimality, agrees with M_*. We write the initial point (ω,y,z) as one of those limits in order to conclude that $y-z\in X_+-\operatorname{Int}X_+$, a contradiction.

We now take $(\tilde{\omega}, \tilde{y}, \tilde{z}) \in M_*$ with $\tilde{z} < \tilde{y}$, where $\tilde{\omega}$ is the point appearing in (h4*), and apply this property to ensure that the point $(\tilde{\omega} \cdot \tilde{t}, u(\tilde{t}, \tilde{\omega}, \tilde{y}), u(\tilde{t}, \tilde{\omega}, \tilde{z}))$ of M_* satisfies $u(\tilde{t}, \tilde{\omega}, \tilde{z}) \ll u(\tilde{t}, \tilde{\omega}, \tilde{y})$, which as seen above is impossible. Note that (h4*) applies here: $M = \{(\omega, z) \mid \exists(\omega, y, z) \in M_*\}$ is a τ-minimal set and there exists a backward τ^K-orbit of $(\tilde{\omega}, \tilde{y}, \tilde{z})$ in M_* and this set is below the copy of the base K_*, which ensures the existence of ordered backward τ-orbits of the points $(\tilde{\omega}, \tilde{z})$ in M and $(\tilde{\omega}, \tilde{y})$ in K. The conclusion is that such a set M_* cannot exist, as asserted.

Let us now deduce that there is not any minimal set strictly below K. Let $M \leqslant K$ be a minimal set. We take $(\omega, z) \in M$ and $(\omega, y) \in K$ with $z \leqslant y$. Since the τ^K-semiorbit of (ω, y, z) is bounded, given a point $(\omega, y_1, y_1) \in K_*$ there is $(t_n) \uparrow \infty$ with $\lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, y), u(t_n, \omega, z)) = (\omega, y_1, y_1)$. This means that $(\omega, y_1) \in M \cap K$, so that both sets coincide, as asserted.

Our next step is to show the absence of minimal subsets of X_+^K different from K. We define

$$\widetilde{\Omega}_0^K = \Big\{\omega \in \widetilde{\Omega}_b^K \;\Big|\; \exists(\omega, y) \in K,\ e \gg 0 \text{ with the semiorbit of } (\omega, y - e)$$

$$\text{bounded and } \liminf_{t\to\infty}\big\|u(t, \omega, y) - u(t, \omega, y - e)\big\| = 0\Big\}.$$

Let us check that $\widetilde{\Omega}_0^K = \widetilde{\Omega}_b^K$. Assume the existence of $\omega \in \widetilde{\Omega}_b^K - \widetilde{\Omega}_0^K$, and let $(\omega, y - e)$ with $(\omega, y) \in K$ and $e \gg 0$ have bounded semiorbit. We fix $(\tilde{\omega}, \tilde{y}) \in K$ with $\tilde{\omega}$ appearing in (h4*), and write it as $\lim_{n\to\infty}(\omega \cdot t_n, u(t_n, \omega, y - e))$ in such a way that there exists $y^* = \lim_{n\to\infty} u(t_n, \omega, y)$. Then $\tilde{y} < y^*$, since otherwise $\omega \in \widetilde{\Omega}_0^K$. Applying (h4*) we conclude that $u(\tilde{t}, \tilde{\omega}, \tilde{y}) \ll u(\tilde{t}, \tilde{\omega}, y^*)$. Note that there exist ordered backward orbits of $(\tilde{\omega}, \tilde{y})$ and $(\tilde{\omega}, y^*)$ in K, given for each $s < 0$ by the limits of $(\omega \cdot (t_m - s), u(t_m - s, \omega, y - e))$ and $(\omega \cdot (t_{\tilde{m}} - s), u(t_{\tilde{m}} - s, \omega, y))$ for suitable subsequences (t_m) of (t_n) and $(t_{\tilde{m}})$ of (t_m). The conclusion is that for any $y_1 \in K_{\tilde{\omega}\cdot\tilde{t}}$ there is $y_2 \in K_{\tilde{\omega}\cdot\tilde{t}}$ such that $y_1 \ll y_2$. But this is impossible, since Zorn's lemma ensures the existence of maximal elements for the order in $K_{\tilde{\omega}\cdot\tilde{t}}$.

Consequently, there is at least one point $\omega \in \widetilde{\Omega}_0^K$. Let the semiorbit of $(\omega, y - e)$ be bounded for $(\omega, y) \in K$ and $e \gg 0$ with $\lim_{n\to\infty} \|u(t_n, \omega, y) - u(t_n, \omega, y - e)\| = 0$ for a suitable $(t_n) \uparrow \infty$. By concavity and monotonicity, $u(t_n, \omega, y) - u(t_n, \omega, y - e) \geqslant u(t_n, \omega, y + e) - u(t_n, \omega, y) \geqslant 0$, so that $\liminf_{t\to\infty} \|u(t, \omega, y + e) - u(t, \omega, y)\| = 0$. The step is completed.

Note that the previous property ensures that Case A of Theorem 4.2 does not hold and that Proposition 4.4(ii) guarantees that Ω_b^K is nonempty. Hence, Theorem 4.3(ii) ensures that any other minimal M satisfies $M \leqslant K$, which, as seen above, ensures that $M = K$. □

Corollary 4.6. *Assume that the semiflow satisfies hypotheses* (h1), (h2), (h3*) *and* (h4*), *that the restriction of the semiflow to X_-^K is global, and that the dynamics on X_+^K fits Case* C *of Theorem* 4.2. *Then K is the unique minimal set.*

Proof. The result follows from Theorem 4.5 if $\widetilde{\Omega}_b^K$ is nonempty, since its condition in (i) precludes Case C. Now assume that $\widetilde{\Omega}_b^K$ is empty and, by contradiction, the existence of a minimal set M different from K, which by Theorem 4.3 satisfies $M < K$. We take $(\tilde{\omega}, y) \in K$ and $(\tilde{\omega}, x) \in M$ with $x < y$, and write $(\tilde{\omega}, y) = \lim_{n\to\infty}(\tilde{\omega} \cdot t_n, u(t_n, \tilde{\omega}, y))$ for a suitable $(t_n) \uparrow \infty$ such that there exists $x_1 = \lim_{n\to\infty} u(t_n, \tilde{\omega}, x)$. Then $x_1 < y$ and the points $(\tilde{\omega}, y)$ and $(\tilde{\omega}, x_1)$ admit ordered backward orbits in K and M respectively. Property (h4*) ensures that $u(\tilde{t}, \tilde{\omega}, x_1) \ll u(\tilde{t}, \tilde{\omega}, y)$, which in turn ensures that $\tilde{\omega} \cdot \tilde{t} \in \widetilde{\Omega}_b^K$. □

So that, in the case that $\widetilde{\Omega}_b^K$ is nonempty and (h4*) holds, Case A can only occur when an additional condition on uniform separation of semiorbits holds for a particular point. If no such a point exists, the dynamics above K may fit Case B (for instance, if $\widetilde{\Omega}_b^K = \Omega$) or C. We finally point out that the above results are optimal, in the sense that all the described possibilities occur for well-known examples.

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